

The universal DAHA of type (C_1^\vee, C_1) and Leonard pairs of q -Racah type

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Abstract

A Leonard pair is a pair of diagonalizable linear transformations of a finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other one. Let \mathbb{F} denote an algebraically closed field, and fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. The universal double affine Hecke algebra (DAHA) \hat{H}_q of type (C_1^\vee, C_1) is the associative \mathbb{F} -algebra defined by generators $\{t_i^{\pm 1}\}_{i=0}^3$ and relations (i) $t_i t_i^{-1} = t_i^{-1} t_i = 1$; (ii) $t_i + t_i^{-1}$ is central; (iii) $t_0 t_1 t_2 t_3 = q^{-1}$. We consider the elements $X = t_3 t_0$ and $Y = t_0 t_1$ of \hat{H}_q . Let \mathcal{V} denote a finite-dimensional irreducible \hat{H}_q -module on which each of X, Y is diagonalizable and t_0 has two distinct eigenvalues. Then \mathcal{V} is a direct sum of the two eigenspaces of t_0 . We show that the pair $X + X^{-1}, Y + Y^{-1}$ acts on each eigenspace as a Leonard pair, and each of these Leonard pairs falls into a class said to have q -Racah type. Thus from \mathcal{V} we obtain a pair of Leonard pairs of q -Racah type. It is known that a Leonard pair of q -Racah type is determined up to isomorphism by a parameter sequence (a, b, c, d) called its Huang data. Given a pair of Leonard pairs of q -Racah type, we find necessary and sufficient conditions on their Huang data for that pair to come from the above construction.

1 Introduction

Throughout the paper \mathbb{F} denotes an algebraically closed field. Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. An \mathbb{F} -algebra is meant to be associative and have a 1.

We begin by recalling the notion of a Leonard pair. We use the following terms. A square matrix is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Definition 1.1 (See [12, Definition 1.1].) Let V denote a vector space over \mathbb{F} with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair of \mathbb{F} -linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ that satisfy (i) and (ii) below:

- (i) there exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal;
- (ii) there exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We say that A, A^* is *over* \mathbb{F} . By the *diameter* of A, A^* we mean $\dim V - 1$.

Note 1.2 According to a common notational convention, A^* denotes the conjugate-transpose of A . We are not using this convention. In a Leonard pair A, A^* the \mathbb{F} -linear transformations A and A^* are arbitrary subject to (i) and (ii) above.

We refer the reader to [2, 3, 8, 12–14] for background on Leonard pairs.

We recall the notion of an isomorphism of Leonard pairs. Let V (resp. V') denote a vector space over \mathbb{F} with finite positive dimension, and let A, A^* (resp. A', A'^*) denote a Leonard pair on V (resp. V'). By an *isomorphism of Leonard pairs* from A, A^* to A', A'^* we mean an \mathbb{F} -linear bijection $f : V \rightarrow V'$ such that both $fA = A'f$ and $fA^* = A'^*f$.

We recall some facts about the eigenvalues of a Leonard pair. We use the following terms. Let V denote a vector space over \mathbb{F} with finite positive dimension and let $A : V \rightarrow V$ denote an \mathbb{F} -linear transformation. Then A is said to be *diagonalizable* whenever V is spanned by the eigenspaces of A . We say A is *multiplicity-free* whenever A is diagonalizable and each eigenspace of A has dimension one. Let A, A^* denote a Leonard pair on V . Then each of A, A^* is multiplicity-free (see [12, Lemma 1.3]). Let $\{\theta_r\}_{r=0}^d$ denote an ordering of the eigenvalues of A . For $0 \leq r \leq d$ let $0 \neq v_r \in V$ denote an eigenvector of A associated with θ_r . Observe that $\{v_r\}_{r=0}^d$ is a basis for V . The ordering $\{\theta_r\}_{r=0}^d$ is said to be *standard* whenever the basis $\{v_r\}_{r=0}^d$ satisfies Definition 1.1(ii). A standard ordering of the eigenvalues of A^* is similarly defined. Let $\{\theta_r\}_{r=0}^d$ denote a standard ordering of the eigenvalues of A . Then the ordering $\{\theta_{d-r}\}_{r=0}^d$ is also standard and no further ordering is standard. A similar result applies to A^* .

Definition 1.3 Let $d \geq 0$ denote an integer and let $\{\theta_r\}_{r=0}^d$ denote a sequence of scalars in \mathbb{F} . The sequence $\{\theta_r\}_{r=0}^d$ is said to be *q-Racah* whenever there exists a nonzero $\alpha \in \mathbb{F}$ such that $\theta_r = \alpha q^{2r-d} + \alpha^{-1} q^{d-2r}$ for $0 \leq r \leq d$. The scalar α is uniquely determined by $\{\theta_r\}_{r=0}^d$ if $d \geq 1$, and determined up to inverse if $d = 0$. We call α the *parameter* of the *q-Racah* sequence.

For a *q-Racah* sequence $\{\theta_r\}_{r=0}^d$ with parameter α , the inverted sequence $\{\theta_{d-r}\}_{r=0}^d$ is *q-Racah* with parameter α^{-1} .

Definition 1.4 Let A, A^* denote a Leonard pair over \mathbb{F} with diameter d . Then A, A^* is said to have *q-Racah type* whenever for each of A, A^* a standard ordering of the eigenvalues forms a *q-Racah* sequence.

Referring to Definition 1.4, assume that A, A^* has *q-Racah type*. Let $\{\theta_r\}_{r=0}^d$ (resp. $\{\theta_r^*\}_{r=0}^d$) denote a standard ordering of the eigenvalues of A (resp. A^*). Let a (resp. b) denote the parameter of the *q-Racah* sequence $\{\theta_r\}_{r=0}^d$ (resp. $\{\theta_r^*\}_{r=0}^d$). It is known that A, A^* is determined up to isomorphism by a, b, d and one more nonzero scalar $c \in \mathbb{F}$. The sequence (a, b, c, d) is called a *Huang data* of A, A^* . The scalar c is defined up to inverse if $d \geq 1$, and arbitrary if $d = 0$ (see Section 2). For a Huang data (a, b, c, d) of A, A^* , each of $(a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d)$ is a Huang data of A, A^* . Moreover A, A^* has no further Huang data, provided that $d \geq 1$.

Next we recall the universal double affine Hecke algebra \hat{H}_q . The double affine Hecke algebra (DAHA) was introduced by Cherednik [1]. The DAHA of type (C_1^\vee, C_1) was studied

by Macdonald [7, Ch. 6], Noumi-Stokman [9], Sahi [10, 11], Koornwinder [5, 6], and Ito-Terwilliger [4]. The algebra \hat{H}_q was introduced by the second author as a generalization of the DAHA of type (C_1^\vee, C_1) . We now recall the definition of \hat{H}_q . For notational convenience define $\mathbb{I} = \{0, 1, 2, 3\}$.

Definition 1.5 (See [16, Definition 3.1].) Let \hat{H}_q denote the \mathbb{F} -algebra defined by generators $\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$ and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad i \in \mathbb{I}, \quad (1)$$

$$t_i + t_i^{-1} \text{ is central} \quad i \in \mathbb{I}, \quad (2)$$

$$t_0 t_1 t_2 t_3 = q^{-1}. \quad (3)$$

The algebra \hat{H}_q is called the *universal DAHA of type (C_1^\vee, C_1)* .

Referring to Definition 1.5, for $i \in \mathbb{I}$ define

$$T_i = t_i + t_i^{-1}.$$

Note that T_i is central in \hat{H}_q .

Definition 1.6 Let \mathcal{V} denote a finite-dimensional irreducible \hat{H}_q -module. By Schur's lemma each T_i acts on \mathcal{V} as a scalar. Write this scalar as $k_i + k_i^{-1}$ with $0 \neq k_i \in \mathbb{F}$. Thus

$$T_i = k_i + k_i^{-1} \quad \text{on } \mathcal{V}. \quad (4)$$

We refer to $\{k_i\}_{i \in \mathbb{I}}$ as a *parameter sequence* of \mathcal{V} .

Referring to Definition 1.6, note that each k_i is defined up to inverse. So each of the 16 sequences $\{k_i^{\pm 1}\}_{i \in \mathbb{I}}$ is a parameter sequence of \mathcal{V} , and \mathcal{V} has no further parameter sequence. Observe by (4) that $(t_i - k_i)(t_i - k_i^{-1})\mathcal{V} = 0$. Thus the eigenvalues of t_i are among k_i, k_i^{-1} .

We consider the following elements of \hat{H}_q :

$$X = t_3 t_0, \quad Y = t_0 t_1, \quad \mathbb{A} = Y + Y^{-1}, \quad \mathbb{B} = X + X^{-1}. \quad (5)$$

It is known that each of \mathbb{A}, \mathbb{B} commutes with t_0 (see Lemma 3.6). By an *XD* (resp. *YD*) \hat{H}_q -module we mean a finite-dimensional irreducible \hat{H}_q -module on which X (resp. Y) is diagonalizable. An \hat{H}_q -module is said to be *feasible* whenever (i) it is both XD and YD; (ii) t_0 has two distinct eigenvalues. Let \mathcal{V} denote a feasible \hat{H}_q -module with parameter sequence $\{k_i\}_{i \in \mathbb{I}}$. Then $k_0^2 \neq 1$. Moreover t_0 is diagonalizable on \mathcal{V} with eigenvalues k_0 and k_0^{-1} . Observe that \mathcal{V} is a direct sum of the two eigenspaces of t_0 , and each eigenspace is invariant under \mathbb{A}, \mathbb{B} . We remark that t_0 does not commute with X, Y , and so the eigenspaces of t_0 may not be invariant under X, Y .

We now state our first main result.

Theorem 1.7 *Let \mathcal{V} denote a feasible \hat{H}_q -module. Then \mathbb{A}, \mathbb{B} act on each eigenspace of t_0 as a Leonard pair of q -Racah type.*

Let \mathcal{V} denote a feasible \hat{H}_q -module. By Theorem 1.7 we obtain a pair of Leonard pairs of q -Racah type. In order to describe how these Leonard pairs are related, we use the following notion. Let A, A^* denote a Leonard pair on V and let $A', A^{*'}$ denote a Leonard pair on V' . We say that these Leonard pairs are *linked* whenever the direct sum $V \oplus V'$ supports a feasible \hat{H}_q -module structure such that V, V' are the eigenspaces of t_0 and

$$\mathbb{A}|_V = A, \quad \mathbb{B}|_V = A^*, \quad \mathbb{A}|_{V'} = A', \quad \mathbb{B}|_{V'} = A^{*'}. \quad (6)$$

We now state our second main result.

Theorem 1.8 *Suppose we are given two Leonard pairs A, A^* and $A', A^{*'}$ over \mathbb{F} that have q -Racah type. Then these Leonard pairs are linked if and only if there exist a Huang data (a, b, c, d) of A, A^* and a Huang data (a', b', c', d') of $A', A^{*'}$ that satisfy one of (i)–(vii) below:*

Case	$d' - d$	a'/a	b'/b	c'/c	Inequalities		
(i)	-2	1	1	1			
(ii)	-1	q	q	q	$a^2 \neq q^{-2d}$	$b^2 \neq q^{-2d}$	
(iii)	0	q^2	1	1		$b^2 \neq q^{\pm 2d}$	$a^2 \neq q^{-2}$
(iv)	0	1	q^2	1	$a^2 \neq q^{\pm 2d}$		$b^2 \neq q^{-2}$
(v)	0	1	1	q^2	$a^2 \neq q^{\pm 2d}$	$b^2 \neq q^{\pm 2d}$	$c^2 \neq q^{-2}$
(vi)	1	q^{-1}	q^{-1}	q^{-1}	$a^2 \neq q^{-2d}$	$b^2 \neq q^{-2d}$	
(vii)	2	1	1	1			

Remark 1.9 Referring to Theorem 1.8, in each of (ii)–(vi) there appear some inequalities. For each of these inequalities we explain the role in our construction of a feasible \hat{H}_q -module. The inequalities in the first column are needed to make Y diagonalizable. The inequalities in the second column are needed to make X diagonalizable. The inequalities in the third column are needed to make t_0 have two distinct eigenvalues.

Remark 1.10 Referring to Theorem 1.8, for $d \geq 2$ we have $a^2 \neq q^{-2}$ and $b^2 \neq q^{-2}$ by Lemma 2.8(i), so these inequalities can be deleted from the table.

Remark 1.11 Suppose we exchange our two Leonard pairs in Theorem 1.8. In the conditions (i)–(vii), the Huang data (a, b, c, d) and (a', b', c', d') are exchanged. After this exchange, the conditions (i), (ii), (vi), (vii) become the original conditions (vii), (vi), (ii), (i) respectively. Concerning the conditions (iii)–(v), we replace each of a, b, c, a', b', c' with its inverse after the above exchange. Then the conditions (iii)–(v) become the original conditions (iii)–(v) respectively.

In a moment we will summarize the proof of Theorems 1.7 and 1.8. Prior to that we explain the significance of the cases that show up in Theorem 1.8. Let \mathcal{V} denote an XD

Similarly, for the reduced X -diagram SS one of the following cases occurs (see Lemma 8.3):

Case	Eigenvalues of t_1 on $\mathcal{V}_X(\mu_0)$ and $\mathcal{V}_X(\mu_n)$	Eigenvalues of t_2 on $\mathcal{V}_X(\mu_0)$ and $\mathcal{V}_X(\mu_n)$
SSa	same	reciprocals
SSb	reciprocals	same

(11)

We refer to each case DS, DDa, DDb, SSa, SSb as the X -type of \mathcal{V} . An XD \hat{H}_q -module is determined up to isomorphism by its dimension, its parameter sequence, and its X -type (see Note 9.3). The cases (i)–(v) in Theorem 1.8 correspond to the X -types as follows:

(i)	(ii)	(iii)	(iv)	(v)
DDa	DS	SSa	DDb	SSb

(12)

The cases (vi) and (vii) are reduced to (ii) and (i) by exchanging our two Leonard pairs (see Remark 1.11).

Recall that \mathcal{V} has 16 parameter sequences $\{k_i^{\pm 1}\}_{i \in \mathbb{I}}$. In view of (9)–(11) we adopt the following convention for most of the paper:

Case	Rule
DS	k_0 (resp. k_3) is the eigenvalue of t_0 (resp. t_3) on $\mathcal{V}_X(\mu_0)$ k_1 (resp. k_2) is the eigenvalue of t_1 (resp. t_2) on $\mathcal{V}_X(\mu_n)$
DD	k_0 (resp. k_3) is the eigenvalue of t_0 (resp. t_3) on $\mathcal{V}_X(\mu_0)$
SS	k_1 (resp. k_2) is the eigenvalue of t_1 (resp. t_2) on $\mathcal{V}_X(\mu_0)$

(13)

Under this convention, the following equation holds (see Lemma 12.1):

X -type of \mathcal{V}	Equation
DS	$k_0 k_1 k_2 k_3 = q^{-n-1}$
DDa	$k_0^2 = q^{-n-1}$
DDb	$k_3^2 = q^{-n-1}$
SSa	$k_1^2 = q^{-n-1}$
SSb	$k_2^2 = q^{-n-1}$

(14)

Below we summarize our proof of the main theorems. Until starting the proof summary of Theorem 1.8, the following notation is in effect. Let \mathcal{V} denote an XD \hat{H}_q -module with dimension $n + 1$, $n \geq 1$. We consider the reduced X -diagram of \mathcal{V} . Let $\{\mu_r\}_{r=0}^n$ denote the eigenvalues of X on \mathcal{V} , as shown in (8). Choose a parameter sequence $\{k_i\}_{i \in \mathbb{I}}$ of \mathcal{V} that satisfies (13). Assume that $k_0 \neq k_0^{-1}$. Let $\mathcal{V}(k_0)$ denote the subspace of \mathcal{V} consisting of $v \in \mathcal{V}$ such that $t_0 v = k_0 v$. The subspace $\mathcal{V}(k_0^{-1})$ is similarly defined.

Towards Theorem 1.7, we make the following observations (A)–(C).

(A) The eigenvalues $\{\mu_r\}_{r=0}^n$ are determined by $\{k_i\}_{i \in \mathbb{I}}$, using (13) and the shape of the diagram.

(B) We indicated earlier that for each single bond μ, ν the subspace $\mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$ is invariant under t_0 . It turns out that the intersections of this subspace with $\mathcal{V}(k_0)$ and

$\mathcal{V}(k_0^{-1})$ each have dimension one. Call these intersections *bond subspaces*. For an endvertex μ that is incident to a double bond, by (9) $\mathcal{V}_X(\mu)$ is invariant under t_0 , so it is contained in one of $\mathcal{V}(k_0)$, $\mathcal{V}(k_0^{-1})$. Call $\mathcal{V}_X(\mu)$ a *bond subspace*. Note that each of $\mathcal{V}(k_0)$, $\mathcal{V}(k_0^{-1})$ is a direct sum of its bond subspaces. Also note that t_0 has only one eigenvalue on \mathcal{V} if and only if $n = 1$ and \mathcal{V} has X -type DDa.

(C) For each vertex μ the subspace $\mathcal{V}_X(\mu)$ is an eigenspace of X with eigenvalue μ . Since $\mathbb{B} = X + X^{-1}$, $\mathcal{V}_X(\mu)$ is invariant under \mathbb{B} with eigenvalue $\mu + \mu^{-1}$. For each single bond μ, ν we have $\mu\nu = 1$, so on $\mathcal{V}_X(\mu)$ and $\mathcal{V}_X(\nu)$ the eigenvalue of \mathbb{B} is the same. Therefore \mathbb{B} acts on $\mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$ as $\mu + \nu$ times the identity.

We summarize our proof of Theorem 1.7. Assume that \mathcal{V} is feasible. First consider the action of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$. We pick a nonzero vector in each bond subspace of $\mathcal{V}(k_0)$. By (B) these vectors form a basis for $\mathcal{V}(k_0)$. We order these vectors along with the ordering $\{\mu_r\}_{r=0}^n$, and denote them by $\{w_r\}_{r=0}^d$. By (C) the vectors $\{w_r\}_{r=0}^d$ are eigenvectors of \mathbb{B} . By (A) the corresponding eigenvalues are represented in terms of $\{k_i\}_{i \in \mathbb{I}}$, and we find that these eigenvalues form a q -Racah sequence. We indicated earlier that for each single (resp. double) bond μ, ν the subspace $\mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$ is invariant under t_0 (resp. t_1). By this we see that the matrix representing \mathbb{A} with respect to $\{w_r\}_{r=0}^d$ is tridiagonal, and it turns out that this tridiagonal matrix is irreducible. We have described the action of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$, and the action of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0^{-1})$ is similar. So far for each of $\mathcal{V}(k_0^{\pm 1})$ there exists a basis with respect to which the matrix representing \mathbb{A} is irreducible tridiagonal and the matrix representing \mathbb{B} is diagonal whose diagonal entries form a q -Racah sequence. There is an automorphism σ of \hat{H}_q that fixes t_0 and swaps \mathbb{A}, \mathbb{B} . Applying the above fact to the twisted \hat{H}_q -module \mathcal{V}^σ , we find that for each of $\mathcal{V}(k_0^{\pm 1})$ there exists a basis with respect to which the matrix representing \mathbb{B} is irreducible tridiagonal and the matrix representing \mathbb{A} is diagonal whose diagonal entries form a q -Racah sequence. Therefore \mathbb{A}, \mathbb{B} act on each $\mathcal{V}(k_0^{\pm 1})$ as a Leonard pair of q -Racah type.

Towards Theorem 1.8, we do the following (D)–(G).

(D) We construct a certain basis $\{u_r\}_{r=0}^n$ for \mathcal{V} such that for $1 \leq r \leq n$ we have $Y^e u_{r-1} \in \mathbb{F}(u_{r-1} - u_r)$, where $e = 1$ if μ_{r-1}, μ_r are 1-adjacent, and $e = -1$ if μ_{r-1}, μ_r are q -adjacent.

(E) We obtain the action of $X^{\pm 1}$ and $Y^{\pm 1}$ on the basis $\{u_r\}_{r=0}^n$. The results show that with respect to the basis $\{u_r\}_{r=0}^n$ the matrices representing $X^{\pm 1}, \mathbb{B}$ are lower tridiagonal, and the matrices representing $Y^{\pm 1}, \mathbb{A}$ are upper tridiagonal.

(F) We obtain some inequalities for $\{k_i\}_{i \in \mathbb{I}}$ from the facts that X is diagonalizable on \mathcal{V} , $\{\mu_r\}_{r=0}^n$ are mutually distinct, and each $\mathcal{V}_X(\mu_r)$ has dimension one (see Lemma 12.4). Also, using the basis $\{u_r\}_{r=0}^n$ we obtain necessary and sufficient conditions on $\{k_i\}_{i \in \mathbb{I}}$ for Y to be diagonalizable on \mathcal{V} (see Corollary 14.8).

(G) Assume that \mathcal{V} is feasible, and consider the two Leonard pairs of q -Racah type from Theorem 1.7. We represent a Huang data of each Leonard pair in terms of $\{k_i\}_{i \in \mathbb{I}}$ as follows. First consider the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$. Using the basis $\{u_r\}_{r=0}^n$ from (D) we construct a basis for $\mathcal{V}(k_0)$ with respect to which the matrix representing \mathbb{A} (resp. \mathbb{B}) is lower bidiagonal (resp. upper bidiagonal). By construction the diagonal entries of this matrix give an ordering of the eigenvalues of \mathbb{A} (resp. \mathbb{B}) on $\mathcal{V}(k_0)$, and it turns out (see Lemma 2.4) that this ordering is standard. This standard ordering forms a q -Racah

sequence. Let a (resp. b) denote the parameter of this q -Racah sequence. We represent a (resp. b) in terms of $\{k_i\}_{i \in \mathbb{I}}$. We remark that the above standard ordering of the eigenvalues of \mathbb{B} coincides with the one from the proof summary of Theorem 1.7 (see Proposition 11.12 and Corollary 19.6). We define a certain nonzero $c \in \mathbb{F}$ in terms of $\{k_i\}_{i \in \mathbb{I}}$. Using the equitable Askey-Wilson relations (see Lemma 20.1), we show that (a, b, c, d) is a Huang data of the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$. So far, we have represented a Huang data of the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ in terms of $\{k_i\}_{i \in \mathbb{I}}$. We similarly represent a Huang data of the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0^{-1})$ in terms of $\{k_i\}_{i \in \mathbb{I}}$.

We now summarize our proof of Theorem 1.8. First consider the “only if” direction. Suppose we are given two Leonard pairs A, A^* on V and A', A'^* on V' that have q -Racah type. Assume that these Leonard pairs are linked. So we have a feasible \hat{H}_q -module structure on $\mathcal{V} := V \oplus V'$ such that V, V' are the eigenspaces of t_0 and (6) holds. Let $\{k_i\}_{i \in \mathbb{I}}$ denote a parameter sequence of \mathcal{V} that satisfies (13). First assume that $V = \mathcal{V}(k_0)$ and $V' = \mathcal{V}(k_0^{-1})$. In (G) we described a Huang data (a, b, c, d) of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ and a Huang data (a', b', c', d') of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0^{-1})$. These Huang data are displayed in Proposition 21.1. For these Huang data the value of $d' - d$ and the ratios $a'/a, b'/b, c'/c$ are as follows.

X -type of \mathcal{V}	$d' - d$	a'/a	b'/b	c'/c
DS	-1	q	q	q
DDa	-2	1	1	1
DDb	0	1	q^2	1
SSa	0	q^2	1	1
SSb	0	1	1	q^2

Using (F), (14) and $k_0^2 \neq 1$ we get the following inequalities:

X -type of \mathcal{V}	inequalities	
DS	$a^2 \neq q^{-2d}$	$b^2 \neq q^{-2d}$
DDa		
DDb	$a^2 \neq q^{\pm 2d}$	$b^2 \neq q^{-2}$
SSa	$b^2 \neq q^{\pm 2d}$	$a^2 \neq q^{-2}$
SSb	$a^2 \neq q^{\pm 2d}$	$b^2 \neq q^{\pm 2d}$ $c^2 \neq q^{-2}$

Now we find that the following case in Theorem 1.8 occurs:

X -type of \mathcal{V}	DS	DDa	DDb	SSa	SSb
Case	(ii)	(i)	(iv)	(iii)	(v)

For the case $V = \mathcal{V}(k_0^{-1})$ and $V' = \mathcal{V}(k_0)$ the argument is similar.

Next consider the “if” direction. Suppose we are given two Leonard pairs A, A^* on V and A', A'^* on V' that have q -Racah type. Assume that there exist a Huang data (a, b, c, d) of A, A^* and a Huang data (a', b', c', d') of A', A'^* that satisfy one of the conditions (i)–(vii). We may assume that the Huang data satisfy one of (i)–(v) by exchanging our two Leonard

pairs if necessary. For each case (i)–(v), we define scalars $\{k_i\}_{i \in \mathbb{I}}$ as follows:

Case	k_0	k_1	k_2	k_3
(i)	q^{-d}	a	c	b
(ii)	$(abcq^{1-d})^{1/2}$	$aq^{-d}k_0^{-1}$	$cq^{-d}k_0^{-1}$	$bq^{-d}k_0^{-1}$
(iii)	aq	q^{-d-1}	b	c
(iv)	bq	c	a	q^{-d-1}
(v)	cq	b	q^{-d-1}	a

(15)

In case (ii), take any one of the square roots of $abcq^{1-d}$ for the value of k_0 . We construct an XD \hat{H}_q -module \mathcal{V} with dimension $d + d' + 1$ and parameter sequence $\{k_i\}_{i \in \mathbb{I}}$ in the following way. Set $n = d + d' + 1$ and let \mathcal{V} denote a vector space over \mathbb{F} with basis $\{v_r\}_{r=0}^n$. We define scalars $\{\mu_r\}_{r=0}^n$ as follows:

Case	Definition of μ_r
(i), (ii), (iv)	$\mu_r = k_0 k_3 q^r$ if r is even, $\mu_r = (k_0 k_3 q^{r+1})^{-1}$ if r is odd
(iii), (v)	$\mu_r = (k_1 k_2 q^{r+1})^{-1}$ if r is even, $\mu_r = k_1 k_2 q^r$ if r is odd

We find that the reduced diagram of $\{\mu_r\}_{r=0}^n$ is as in (8) with

(i)	(ii)	(iii)	(iv)	(v)
DD	DS	SS	DD	SS

Using this diagram we define the action of $\{t_i\}_{i \in \mathbb{I}}$ on $\{v_r\}_{r=0}^n$ as follows. For $0 \leq r \leq n-1$ such that μ_r, μ_{r+1} are 1-adjacent (resp. q -adjacent), we define the action of t_0, t_3 (resp. t_1, t_2) on $\mathbb{F}v_r + \mathbb{F}v_{r+1}$ by (46)–(49) (resp. (50)–(53)). The remaining actions are defined by (43). It turns out that these actions give an \hat{H}_q -module structure of \mathcal{V} . Moreover, this \hat{H}_q -module \mathcal{V} is XD and has X -type

(i)	(ii)	(iii)	(iv)	(v)
DDa	DS	SSa	DDb	SSb

(16)

The inequalities in Theorem 1.8 yield some inequalities for $\{k_i\}_{i \in \mathbb{I}}$ (see Lemma 24.3). By these inequalities we find that $\{k_i\}_{i \in \mathbb{I}}$ satisfy the conditions in (F) that make Y diagonalizable on \mathcal{V} , so \mathcal{V} is YD. We also find that t_0 has two distinct eigenvalues on \mathcal{V} , so \mathcal{V} is feasible. By Theorem 1.7 \mathbb{A}, \mathbb{B} act on each eigenspace of t_0 as a Leonard pair of q -Racah type. These Leonard pairs have Huang data that are displayed in Proposition 21.1. Using (15) we find that these Huang data coincide with (a, b, c, d) and (a', b', c', d') . Thus the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ (resp. $\mathcal{V}(k_0^{-1})$) and the Leonard pair A, A^* (resp. $A', A^{*'}$) have a common Huang data. So the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ (resp. $\mathcal{V}(k_0^{-1})$) and the Leonard pair A, A^* on V (resp. $A', A^{*'}$ on V') are isomorphic. Let $f : \mathcal{V}(k_0) \rightarrow V$ (resp. $f' : \mathcal{V}(k_0^{-1}) \rightarrow V'$) denote the corresponding isomorphism of Leonard pairs. Consider the \mathbb{F} -linear bijection $f \oplus f' : \mathcal{V} = \mathcal{V}(k_0) + \mathcal{V}(k_0^{-1}) \rightarrow V \oplus V'$. We define an \hat{H}_q -module structure on $V \oplus V'$ so that $f \oplus f'$ is an isomorphism of \hat{H}_q -modules. By construction the \hat{H}_q -module $V \oplus V'$ is feasible, the subspaces V and V' are the eigenspaces of t_0 , and (6) holds. Thus the Leonard pairs A, A^* and $A', A^{*'}$ are linked.

The paper is organized as follows. In Section 2 we recall some materials concerning Leonard pairs. In Section 3 we recall some basic facts about \hat{H}_q . In Sections 4 and 5 we consider certain elements of \hat{H}_q and study their properties. In Section 6 we study the X -diagram. In Section 7 we show that X is multiplicity-free on an XD \hat{H}_q -module. In Section 8 we study the cases DS, DDa, DDb, SSa, SSb, and we do (A). In Section 9 we investigate the structure of an XD \hat{H}_q -module. In Section 10 we investigate the eigenspaces of t_0 , and we do (B). In Section 11 we prove Theorem 1.7. In Section 12 we obtain some (in)equalities for $\{k_i\}_{i \in \mathbb{I}}$. In Sections 13 we do (D). In Section 14 we obtain the action of $Y^{\pm 1}$ on the basis $\{u_r\}_{r=0}^n$, and we do (F). In Section 15 we obtain the action of $\{t_i\}_{i \in \mathbb{I}}$ on the basis $\{u_r\}_{r=0}^n$. In Section 16 we obtain the action of $X^{\pm 1}$ on the basis $\{u_r\}_{r=0}^n$. In Sections 17–21 we do (G). In Section 22 we prove the “only if” direction of Theorem 1.8. In Sections 23 and 24 we prove the “if” direction of Theorem 1.8.

2 Preliminaries for Leonard pairs

In this section we recall some materials concerning Leonard pairs. We first recall the notion of a parameter array, and next recall the notion of a Huang data.

Fix an integer $d \geq 0$. Let M denote a $(d+1) \times (d+1)$ matrix with all entries in \mathbb{F} . We index the rows and columns by $0, 1, \dots, d$. Let V denote a vector space over \mathbb{F} with basis $\{v_r\}_{r=0}^d$, and consider a linear transformation $A : V \rightarrow V$. We say M represents A with respect to $\{v_r\}_{r=0}^d$ whenever $Av_s = \sum_{r=0}^d M_{r,s} v_r$ for $0 \leq s \leq d$.

Lemma 2.1 (See [12, Theorem 3.2].) *Let V denote a vector space over \mathbb{F} with dimension $d+1$, and let A, A^* denote a Leonard pair on V . Let $\{\theta_r\}_{r=0}^d$ (resp. $\{\theta_r^*\}_{r=0}^d$) denote a standard ordering of the eigenvalues of A (resp. A^*). Then there exists a basis for V with respect to which the matrices representing A, A^* are*

$$A : \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_d \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_d \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix}$$

for some scalars $\{\varphi_r\}_{r=1}^d$ in \mathbb{F} . The sequence $\{\varphi_r\}_{r=1}^d$ is uniquely determined by the ordering $(\{\theta_r\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d)$. Moreover $\varphi_r \neq 0$ for $1 \leq r \leq d$.

With reference to Lemma 2.1 we refer to $\{\varphi_r\}_{r=1}^d$ as the *first split sequence* of A, A^* associated with the ordering $(\{\theta_r\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d)$. By the *second split sequence* of A, A^* associated with the ordering $(\{\theta_r\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d)$ we mean the first split sequence of A, A^* associate with the ordering $(\{\theta_{d-r}\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d)$.

Definition 2.2 Let A, A^* denote a Leonard pair over \mathbb{F} . By a *parameter array* of A, A^* we mean a sequence

$$(\{\theta_r\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d, \{\varphi_r\}_{r=1}^d, \{\phi_r\}_{r=1}^d), \quad (17)$$

such that $\{\theta_r\}_{r=0}^d$ is a standard ordering of the eigenvalues of A , $\{\theta_r^*\}_{r=0}^d$ is a standard ordering of the eigenvalues of A^* , and $\{\varphi_r\}_{r=1}^d$ (resp. $\{\phi_r\}_{r=1}^d$) is the corresponding first split sequence (resp. second split sequence) of A, A^* .

Let A, A^* denote a Leonard pair over \mathbb{F} with diameter d , and let (17) denote a parameter array of A, A^* . Then by [12, Theorem 1.11] the parameter arrays of A, A^* are as follows:

$$\begin{aligned} &(\{\theta_r\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d, \{\varphi_r\}_{r=1}^d, \{\phi_r\}_{r=1}^d), \\ &(\{\theta_r\}_{r=0}^d, \{\theta_{d-r}^*\}_{r=0}^d, \{\phi_{d-r+1}\}_{r=1}^d, \{\varphi_{d-r+1}\}_{r=1}^d), \\ &(\{\theta_{d-r}\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d, \{\phi_r\}_{r=1}^d, \{\varphi_r\}_{r=1}^d), \\ &(\{\theta_{d-r}\}_{r=0}^d, \{\theta_{d-r}^*\}_{r=0}^d, \{\varphi_{d-r+1}\}_{r=1}^d, \{\phi_{d-r+1}\}_{r=1}^d). \end{aligned}$$

Lemma 2.3 (See [12, Theorem 1.9].) *Consider two Leonard pairs over \mathbb{F} . Assume that these Leonard pairs have a common parameter array. Then these Leonard pairs are isomorphic.*

A square matrix is said to be *upper bidiagonal* whenever each nonzero entry lies on the diagonal or the superdiagonal, and *lower bidiagonal* whenever its transpose is upper bidiagonal.

Lemma 2.4 (See [15, Corollary 7.6].) *Let V denote a vector space over \mathbb{F} with dimension $d+1$, and let A, A^* denote a Leonard pair on V . Assume that there exists a basis for V with respect to which the matrix representing A (resp. A^*) is lower bidiagonal (resp. upper bidiagonal) with (r, r) -entry θ_r (resp. θ_r^*) for $0 \leq r \leq d$. Then $\{\theta_r\}_{r=0}^d$ (resp. $\{\theta_r^*\}_{r=0}^d$) is a standard ordering of the eigenvalues of A (resp. A^*).*

Lemma 2.5 (See [3, Corollary 6.4].) *Let A, A^* denote a Leonard pair over \mathbb{F} with diameter d , and let (17) denote a parameter array of A, A^* . Assume that A, A^* has q -Racah type, and let a, b denote nonzero scalars in \mathbb{F} such that*

$$\theta_r = aq^{2r-d} + a^{-1}q^{d-2r}, \quad \theta_r^* = bq^{2r-d} + b^{-1}q^{d-2r} \quad (0 \leq r \leq d). \quad (18)$$

Then there exists a nonzero $c \in \mathbb{F}$ such that

$$\varphi_r = a^{-1}b^{-1}q^{d+1}(q^r - q^{-r})(q^{r-d-1} - q^{d-r+1})(q^{-r} - abcq^{r-d-1})(q^{-r} - abc^{-1}q^{r-d-1}), \quad (19)$$

$$\phi_r = ab^{-1}q^{d+1}(q^r - q^{-r})(q^{r-d-1} - q^{d-r+1})(q^{-r} - a^{-1}bcq^{r-d-1})(q^{-r} - a^{-1}bc^{-1}q^{r-d-1}) \quad (20)$$

for $1 \leq r \leq d$. Moreover c is unique up to inverse, provided that $d \geq 1$.

With reference to Lemma 2.5, for the case $d = 0$ the conditions (19) and (20) become vacuous, so any nonzero $c \in \mathbb{F}$ satisfies these conditions. For the case $d \geq 1$, the scalars a, b, c are determined up to inverse of c by the ordering $(\{\theta_r\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d)$, since the parameter array (17) is determined by the ordering $(\{\theta_r\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d)$.

Definition 2.6 Let A, A^* denote a Leonard pair over \mathbb{F} with diameter d that has q -Racah type, and let (17) denote a parameter array of A, A^* . Let a, b, c denote nonzero scalars in \mathbb{F} that satisfy (18)–(20). Then the sequence (a, b, c, d) is called a *Huang data* of A, A^* corresponding to the ordering $(\{\theta_r\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d)$.

With reference to Definition 2.6, assume $d \geq 1$. Let (a, b, c, d) denote a Huang data of A, A^* corresponding to the ordering $(\{\theta_r\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d)$. Then by [3, Lemma 8.1] for each standard ordering of the eigenvalues of A, A^* the corresponding Huang data of A, A^* are as follows:

Standard ordering	Huang data
$(\{\theta_r\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d)$	$(a, b, c^{\pm 1}, d)$
$(\{\theta_r\}_{r=0}^d, \{\theta_{d-r}^*\}_{r=0}^d)$	$(a, b^{-1}, c^{\pm 1}, d)$
$(\{\theta_{d-r}\}_{r=0}^d, \{\theta_r^*\}_{r=0}^d)$	$(a^{-1}, b, c^{\pm 1}, d)$
$(\{\theta_{d-r}\}_{r=0}^d, \{\theta_{d-r}^*\}_{r=0}^d)$	$(a^{-1}, b^{-1}, c^{\pm 1}, d)$

Lemma 2.7 Consider two Leonard pairs over \mathbb{F} that have q -Racah type. Assume that these Leonard pairs have a common Huang data. Then these Leonard pairs are isomorphic.

Proof. Let (a, b, c, d) denote a common Huang data of the two Leonard pairs. By (18)–(20) the Huang data (a, b, c, d) determines a parameter array of each Leonard pair. Thus these Leonard pairs have a common parameter array. By this and Lemma 2.3 these Leonard pairs are isomorphic. \square

We mention a lemma for later use.

Lemma 2.8 (See [3, Lemmas 7.2, 7.3].) Let a, b, c denote nonzero scalars in \mathbb{F} . Then the sequence (a, b, c, d) is a Huang data of a Leonard pair over \mathbb{F} of q -Racah type if and only if the following (i) and (ii) hold:

- (i) neither of a^2, b^2 is among $q^{2d-2}, q^{2d-4}, \dots, q^{2-2d}$;
- (ii) none of $abc, a^{-1}bc, ab^{-1}c, abc^{-1}$ is among $q^{d-1}, q^{d-3}, \dots, q^{1-d}$.

3 Basic facts about \hat{H}_q

In this section we collect some basic facts about \hat{H}_q . For more background information we refer the reader to [16]. The following two lemmas are immediate from Definition 1.5.

Lemma 3.1 In the algebra \hat{H}_q the scalar q^{-1} is equal to each of

$$t_0 t_1 t_2 t_3, \quad t_1 t_2 t_3 t_0, \quad t_2 t_3 t_0 t_1, \quad t_3 t_0 t_1 t_2.$$

Lemma 3.2 There exists an automorphism ϱ of \hat{H}_q that sends

$$t_0 \mapsto t_1 \mapsto t_2 \mapsto t_3 \mapsto t_0.$$

Lemma 3.3 *There exists an automorphism σ of \hat{H}_q that sends*

$$t_0 \mapsto t_0, \quad t_1 \mapsto t_0^{-1}t_3t_0, \quad t_2 \mapsto t_1t_2t_1^{-1}, \quad t_3 \mapsto t_1.$$

Proof. This is routinely checked, or see [4, Lemma 4.2]. \square

The following lemmas are routinely checked.

Lemma 3.4 *The automorphism ϱ from Lemma 3.2 sends*

$$X \mapsto Y \mapsto q^{-1}X^{-1} \mapsto q^{-1}Y^{-1} \mapsto X.$$

Lemma 3.5 *The automorphism σ from Lemma 3.3 sends*

$$\begin{array}{llll} X \mapsto t_0^{-1}Yt_0, & Y \mapsto X, & \mathbb{A} \mapsto \mathbb{B}, & \mathbb{B} \mapsto \mathbb{A}, \\ T_0 \mapsto T_0, & T_1 \mapsto T_3, & T_2 \mapsto T_2, & T_3 \mapsto T_1. \end{array}$$

We collect some identities in \hat{H}_q .

Lemma 3.6 (See [16, Corollary 6.2].) *For $i, j \in \mathbb{I}$ the following hold:*

- (i) $t_it_j + (t_it_j)^{-1} = t_jt_i + (t_jt_i)^{-1}$;
- (ii) $t_it_j + (t_it_j)^{-1}$ commutes with each of t_i, t_j .

Lemma 3.7 *We have*

$$Xt_0 - t_0X^{-1} = XT_0 - T_3, \tag{21}$$

$$qXt_2 - q^{-1}t_2X^{-1} = T_1 - q^{-1}X^{-1}T_2. \tag{22}$$

Proof. To verify (21), eliminate T_0 (resp. T_3) using $T_0 = t_0 + t_0^{-1}$ (resp. $T_3 = t_3 + t_3^{-1}$) and simplify the result using $t_3 = Xt_0^{-1}$. To obtain (22), apply ϱ^2 to each side of (21) and use Lemma 3.4 to find

$$q^{-1}X^{-1}t_2 - qt_2X = q^{-1}X^{-1}T_2 - T_1.$$

In this equation, multiply each term on the left by X and on the right by X^{-1} . This yields (22). \square

4 The elements G_i

We consider the following elements of \hat{H}_q that will help us investigate \hat{H}_q -modules.

Definition 4.1 Define

$$\begin{aligned} G_0 &= t_0 - t_3 t_0 t_3^{-1}, & G_1 &= t_1 - t_0 t_1 t_0^{-1}, \\ G_2 &= t_2 - t_1 t_2 t_1^{-1}, & G_3 &= t_3 - t_2 t_3 t_2^{-1}. \end{aligned}$$

In this and the next section, we describe the G_i . We focus on how G_0, G_2 interact with X ; similar results describe how G_1, G_3 interact with Y . Our results are summarized as follows. Let \mathcal{V} denote an XD \hat{H}_q -module with parameter sequence $\{k_i\}_{i \in \mathbb{I}}$. We show that each of G_0^2, G_2^2 acts on \mathcal{V} as a Laurent polynomial in X . In particular G_0^2, G_2^2 act on each X -eigenspace in \mathcal{V} as a scalar multiple of the identity. We compute these scalars in terms of $\{k_i\}_{i \in \mathbb{I}}$.

Lemma 4.2 *We have*

$$\begin{aligned} XG_0 &= G_0X^{-1}, & X^{-1}G_0 &= G_0X, \\ XG_2 &= q^{-2}G_2X^{-1}, & X^{-1}G_2 &= q^2G_2X. \end{aligned}$$

Proof. Using $X = t_3 t_0$ and $G_0 = t_0 - t_3 t_0 t_3^{-1}$ one verifies

$$G_0 t_3 - X^{-1} G_0 X^{-1} t_3 = t_0 t_3 + (t_0 t_3)^{-1} - t_3 t_0 - (t_3 t_0)^{-1}.$$

In this equation, the right-hand side is zero by Lemma 3.6(i). So $G_0 t_3 = X^{-1} G_0 X^{-1} t_3$. In this equation, multiply each side on the left by X and on the right by t_3^{-1} to get $XG_0 = G_0X^{-1}$. In this equation, multiply each side on the left by X^{-1} and on the right by X to get $X^{-1}G_0 = G_0X$. In this equation, apply ϱ^2 to each side and simplify the result using Lemma 3.4 to get $XG_2 = q^{-2}G_2X^{-1}$. In this equation, multiply each side on the left by X^{-1} and on the right by X to get $X^{-1}G_2 = q^2G_2X$. \square

Lemma 4.3 *We have*

$$G_0 = t_0(1 - X^{-2}) + T_3X^{-1} - T_0, \tag{23}$$

$$G_0 = t_3(X^{-1} - X) + T_3X - T_0, \tag{24}$$

$$G_2 = t_2(1 - q^2X^2) + qT_1X - T_2, \tag{25}$$

$$G_2 = t_1(qX - q^{-1}X^{-1}) + q^{-1}T_1X^{-1} - T_2. \tag{26}$$

Proof. In (21), multiply each side on the right by X^{-1} and use $Xt_0X^{-1} = t_0 - G_0$ to get (23). In (23), eliminate t_0 using $t_0 = T_3X - t_3X$ to get (24). Concerning (25), apply ϱ^2 to (23) and use Lemma 3.4. The line (26) is similarly obtained from (24). \square

Proposition 4.4 *We have*

$$G_0^2 = (X + X^{-1})^2 - (X + X^{-1})T_0T_3 + T_0^2 + T_3^2 - 4, \quad (27)$$

$$G_2^2 = (qX + q^{-1}X^{-1})^2 - (qX + q^{-1}X^{-1})T_1T_2 + T_1^2 + T_2^2 - 4. \quad (28)$$

Proof. We first show (27). We claim

$$X^{-1} - X = t_0X^{-1}G_0 + (T_3X^{-1} - T_0)t_3. \quad (29)$$

In (29), represent each side in terms of t_0, t_3, T_0, T_3 using $X = t_3t_0, X^{-1} = t_0^{-1}t_3^{-1}, G_0 = t_0 - t_3t_0t_3^{-1}, t_0^{-1} = T_0 - t_0, t_3^{-1} = T_3 - t_3$, and simplify the results using the fact that T_0, T_3 are central. This gives (29). In (29), multiply each side on the right by $X^{-1} - X$, and simplify the results using Lemma 4.2 to find

$$(X^{-1} - X)^2 = t_0(1 - X^{-2})G_0 + (T_3X^{-1} - T_0)t_3(X^{-1} - X). \quad (30)$$

By (23) $t_0(1 - X^{-2}) = G_0 - (T_3X^{-1} - T_0)$. By (24) $t_3(X^{-1} - X) = G_0 - (T_3X - T_0)$. In (30) eliminate $t_0(1 - X^{-2})$ and $t_3(X^{-1} - X)$ using these comments, and simplify the result to find

$$(X^{-1} - X)^2 = G_0^2 - T_0^2 - T_3^2 + (X + X^{-1})T_0T_3.$$

This implies (27). Concerning (28), apply ϱ^2 to each side of (27) and use Lemma 3.4. \square

5 The action of G_i on the eigenspaces of X

Let \mathcal{V} denote an XD \hat{H}_q -module. In this section we describe how the elements $\{t_i\}_{i \in \mathbb{I}}, G_0, G_2$ act on the eigenspaces of X .

Lemma 5.1 *For $0 \neq \mu \in \mathbb{F}$*

$$G_0\mathcal{V}_X(\mu) \subseteq \mathcal{V}_X(\mu^{-1}), \quad G_2\mathcal{V}_X(\mu) \subseteq \mathcal{V}_X(q^{-2}\mu^{-1}).$$

Proof. We first show $G_0\mathcal{V}_X(\mu) \subseteq \mathcal{V}_X(\mu^{-1})$. Pick any $v \in \mathcal{V}_X(\mu)$. Then $X^{-1}v = \mu^{-1}v$. By Lemma 4.2 $XG_0 = G_0X^{-1}$. Using these comments we argue $XG_0v = G_0X^{-1}v = \mu^{-1}G_0v$. So $G_0v \in \mathcal{V}_X(\mu^{-1})$. We have shown $G_0\mathcal{V}_X(\mu) \subseteq \mathcal{V}_X(\mu^{-1})$. The proof of $G_2\mathcal{V}_X(\mu) \subseteq \mathcal{V}_X(q^{-2}\mu^{-1})$ is similar. \square

For indeterminates λ, s, t define

$$G(\lambda, s, t) = \lambda^{-2}(\lambda - st)(\lambda - st^{-1})(\lambda - s^{-1}t)(\lambda - s^{-1}t^{-1}). \quad (31)$$

The following lemma is routinely verified.

Lemma 5.2 *We have*

$$G(\lambda, s, t) = (\lambda + \lambda^{-1})^2 - (\lambda + \lambda^{-1})(s + s^{-1})(t + t^{-1}) + (s + s^{-1})^2 + (t + t^{-1})^2 - 4.$$

Moreover $G(\lambda, s, t)$ is equal to each of the following:

$$G(\lambda^{-1}, s, t), \quad G(\lambda, s^{-1}, t), \quad G(\lambda, s, t^{-1}).$$

The action of G_0^2, G_2^2 on \mathcal{V} is given as follows. Let $\{k_i\}_{i \in \mathbb{I}}$ denote a parameter sequence of \mathcal{V} .

Lemma 5.3 *The following hold on \mathcal{V} :*

$$G_0^2 = G(X, k_0, k_3), \quad G_2^2 = G(qX, k_1, k_2). \quad (32)$$

Proof. We first show the equation on the left in (32). By Definition 1.6 $T_0 = k_0 + k_0^{-1}$ and $T_3 = k_3 + k_3^{-1}$ on \mathcal{V} . By this and (27)

$$G_0^2 = (X + X^{-1})^2 - (X + X^{-1})(k_0 + k_0^{-1})(k_3 + k_3^{-1}) + (k_0 + k_0^{-1})^2 + (k_3 + k_3^{-1})^2 - 4$$

on \mathcal{V} . In the above line, the the right-hand side is equal to $G(X, k_0, k_3)$ by Lemma 5.2. Thus the equation on the left in (32) holds on \mathcal{V} . The proof of the equation on the right in (32) is similar. \square

The following corollary is immediate from Lemma 5.3.

Corollary 5.4 *Let $0 \neq \mu \in \mathbb{F}$.*

- (i) G_0^2 acts on $\mathcal{V}_X(\mu)$ as $G(\mu, k_0, k_3)$ times the identity.
- (ii) G_2^2 acts on $\mathcal{V}_X(\mu)$ as $G(q\mu, k_1, k_2)$ times the identity.

We mention some lemmas for later use.

Lemma 5.5 *Let μ denote a vertex of the X -diagram of \mathcal{V} that is not incident to a single loop. Then $\mu \neq \mu^{-1}$ and the following hold on $\mathcal{V}_X(\mu)$:*

$$t_0 = \frac{\mu T_0 - T_3 + \mu G_0}{\mu - \mu^{-1}}, \quad t_0^{-1} = \frac{\mu^{-1} T_0 - T_3 + \mu G_0}{\mu^{-1} - \mu}, \quad (33)$$

$$t_3 = \frac{T_0 - \mu T_3 + G_0}{\mu^{-1} - \mu}, \quad t_3^{-1} = \frac{T_0 - \mu^{-1} T_3 + G_0}{\mu - \mu^{-1}}. \quad (34)$$

Proof. We have $\mu \neq \mu^{-1}$ by the definition of a single loop. Now the formula for t_0 follows from (23), and the formula for t_3 follows from (24). Concerning t_0^{-1} and t_3^{-1} , use $t_0^{-1} = T_0 - t_0$ and $t_3^{-1} = T_3 - t_3$. \square

Lemma 5.6 *Let μ denote a vertex of the X -diagram of \mathcal{V} that is not incident to a double loop. Then $q\mu \neq q^{-1}\mu^{-1}$ and the following hold on $\mathcal{V}_X(\mu)$:*

$$t_1 = \frac{q^{-1}\mu^{-1}T_1 - T_2 - G_2}{q^{-1}\mu^{-1} - q\mu}, \quad t_1^{-1} = \frac{q\mu T_1 - T_2 - G_2}{q\mu - q^{-1}\mu^{-1}}, \quad (35)$$

$$t_2 = \frac{T_1 - q^{-1}\mu^{-1}T_2 - q^{-1}\mu^{-1}G_2}{q\mu - q^{-1}\mu^{-1}}, \quad t_2^{-1} = \frac{T_1 - q\mu T_2 - q^{-1}\mu^{-1}G_2}{q^{-1}\mu^{-1} - q\mu}. \quad (36)$$

Proof. Similar to the proof of Lemma 5.5. \square

6 The X -diagram

Let \mathcal{V} denote an XD \hat{H}_q -module. In this section we prove that the reduced X -diagram of \mathcal{V} is a path. We then introduce the notion of a standard ordering of the eigenvalues of X . We also describe some basic facts concerning these notions.

Lemma 6.1 *Let \mathcal{W} denote a subspace of \mathcal{V} that is invariant under X .*

- (i) *\mathcal{W} is invariant under t_0 if and only if \mathcal{W} is invariant under t_3 .*
- (ii) *\mathcal{W} is invariant under t_2 if and only if \mathcal{W} is invariant under t_1 .*
- (iii) *\mathcal{W} is an \hat{H}_q -submodule of \mathcal{V} if and only if \mathcal{W} is invariant under each of t_0, t_2 .*

Proof. We have $X\mathcal{W} \subseteq \mathcal{W}$. This forces $X\mathcal{W} = \mathcal{W}$ since X is invertible and \mathcal{W} is finite-dimensional.

(i): First assume $t_0\mathcal{W} \subseteq \mathcal{W}$. This forces $t_0\mathcal{W} = \mathcal{W}$, and so $t_0^{-1}\mathcal{W} = \mathcal{W}$. By this and $t_3 = Xt_0^{-1}$ we get $t_3\mathcal{W} = \mathcal{W}$. Next assume $t_3\mathcal{W} \subseteq \mathcal{W}$. We get $t_0\mathcal{W} = \mathcal{W}$ in a similar way.

(ii): Similar to the proof of (i) using $t_1 = q^{-1}X^{-1}t_2^{-1}$.

(iii): Follows from (i) and (ii). \square

Lemma 6.2 *Let $\mu, \nu \in \mathbb{F}$.*

- (i) *Assume that μ, ν are 1-adjacent. Then $\mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$ is invariant under t_0 .*
- (ii) *Assume that μ, ν are q -adjacent. Then $\mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$ is invariant under t_2 .*

Proof. (i): For notational convenience set $\mathcal{W} = \mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$. We first show $t_0\mathcal{V}_X(\mu) \subseteq \mathcal{W}$. Pick any $w \in \mathcal{V}_X(\mu)$. We show $t_0w \in \mathcal{W}$. Multiply each side of (21) on the left by $X - \mu$, and apply the result to w . Simplifying the result using $Xw = \mu w$, $X^{-1}w = \nu w$ and the fact that T_0, T_3 commute with X , we find $(X - \mu)(X - \nu)t_0w = 0$. By this and since X is diagonalizable we obtain $t_0w \in \mathcal{W}$. We have shown $t_0\mathcal{V}_X(\mu) \subseteq \mathcal{W}$. Interchanging μ and ν in the above arguments, we obtain $t_0\mathcal{V}_X(\nu) \subseteq \mathcal{W}$. Therefore $t_0\mathcal{W} \subseteq \mathcal{W}$.

(ii): Similar to the proof of (i), but use (22) instead of (21). \square

Consider the X -diagram of \mathcal{V} , which we are not assuming is reduced.

Lemma 6.3 *Let μ denote a vertex of the X -diagram.*

- (i) *Assume that μ is not incident to a single bond. Then $G_0\mathcal{V}_X(\mu) = 0$.*
- (ii) *Assume that μ is not incident to a double bond. Then $G_2\mathcal{V}_X(\mu) = 0$.*

Proof. (i): By Lemma 5.1 $G_0\mathcal{V}_X(\mu) \subseteq \mathcal{V}_X(\mu^{-1})$. We have $\mathcal{V}_X(\mu^{-1}) = 0$ since μ is not incident to a single bond. By these comments $G_0\mathcal{V}_X(\mu) = 0$.

(ii): Similar. \square

Lemma 6.4 *Let μ denote a vertex of the X -diagram and let $v \in \mathcal{V}_X(\mu)$.*

- (i) *Assume that μ is not incident to a single loop. Then $\mathbb{F}v + \mathbb{F}G_0v$ is invariant under t_0 .*
- (ii) *Assume that μ is not incident to a double loop. Then $\mathbb{F}v + \mathbb{F}G_2v$ is invariant under t_2 .*

Proof. (i): We show each of t_0v and t_0G_0v is contained in $\mathbb{F}v + \mathbb{F}G_0v$. Recall that T_i ($i \in \mathbb{I}$) acts on \mathcal{V} as a scalar. Applying (33) to v we find that t_0v is contained in $\mathbb{F}v + \mathbb{F}G_0v$. Concerning t_0G_0v , we may assume $G_0v \neq 0$. By Lemma 5.1 $G_0v \in \mathcal{V}_X(\mu^{-1})$. In particular $\mathcal{V}_X(\mu^{-1}) \neq 0$ and so μ^{-1} is a vertex of the X -diagram. Now applying (33) to G_0v and using Corollary 5.4(i) we find that t_0G_0v is contained in $\mathbb{F}v + \mathbb{F}G_0v$. The result follows.

(ii): Similar to the proof of (i). \square

Lemma 6.5 *Let μ denote a vertex of the X -diagram and let $v \in \mathcal{V}_X(\mu)$.*

- (i) *Assume that μ is not incident to a single bond. Then $\mathbb{F}v$ is invariant under t_0 .*
- (ii) *Assume that μ is not incident to a double bond. Then $\mathbb{F}v$ is invariant under t_2 .*

Proof. (i): By Lemma 6.3(i) $G_0v = 0$. By this and Lemma 6.4(i) $\mathbb{F}v$ is invariant under t_0 .

(ii): Similar. \square

Lemma 6.6 *The X -diagram of \mathcal{V} is connected.*

Proof. Let \mathcal{C} denote a connected component of the X -diagram, and let \mathcal{W} denote the sum of the subspaces $\mathcal{V}_X(\mu)$ for $\mu \in \mathcal{C}$. By Lemmas 6.2 and 6.5 \mathcal{W} is invariant under each of t_0, t_2 . By this and Lemma 6.1(iii) \mathcal{W} is an \hat{H}_q -submodule of \mathcal{V} , and this forces $\mathcal{W} = \mathcal{V}$ since \mathcal{V} is irreducible. Therefore \mathcal{C} contains every vertex, so the X -diagram is connected.

\square

Lemma 6.7 *The reduced X -diagram of \mathcal{V} is a path.*

Proof. By Lemma 6.6 the reduced X -diagram is connected. By the construction, each vertex is 1-adjacent (resp. q -adjacent) to at most one vertex. Moreover there is no cycle in the reduced X -diagram since q is not a root of unity. By these comments the reduced X -diagram is a path. \square

By Lemma 6.7 the reduced X -diagram is one of the types (7). An ordering $\{\mu_r\}_{r=0}^n$ of the eigenvalues of X is said to be *standard* whenever they are attached to the reduced X -diagram as in (8). Assume for the moment that the reduced X -diagram of \mathcal{V} is DD or SS. Let $\{\mu_r\}_{r=0}^n$ denote a standard ordering of the eigenvalues of X . Then the ordering $\{\mu_{n-r}\}_{r=0}^n$ is also standard and no further ordering is standard.

For the rest of this section, we fix a standard ordering $\{\mu_r\}_{r=0}^n$ of the eigenvalues of X . By the shape of the diagram the parity of n is as follows:

X -diagram	Parity of n
DS	even
DD, SS	odd

(37)

By the construction, for $0 \leq r \leq n$ the eigenvalue μ_r is as follows:

X -diagram	μ_r for even r	μ_r for odd r
DS, DD	$q^r \mu_0$	$q^{-r-1} \mu_0^{-1}$
SS	$q^{-r} \mu_0$	$q^{r-1} \mu_0^{-1}$

(38)

In particular, μ_n is as follows:

X -diagram	DS	DD	SS
μ_n	$q^n \mu_0$	$q^{-n-1} \mu_0^{-1}$	$q^{n-1} \mu_0^{-1}$

(39)

Lemma 6.8 *Consider the X -diagram of \mathcal{V} .*

- (i) *Assume $n = 0$. Then μ_0 cannot be incident to both a single loop and a double loop.*
- (ii) *Assume $n \geq 1$. Then at most one of μ_0, μ_n is incident to a loop.*

Proof. First consider the case that the X -diagram \mathcal{V} is DS. By way of contradiction, assume that μ_0 is incident to a single loop and μ_n is incident to a double loop. So $\mu_0^2 = 1$ and $\mu_n^2 = q^{-2}$. By (39) we have $\mu_n = q^n \mu_0$. By these comments $q^{2n+2} = 1$, contradicting the assumption that q is not a root of unity. We have shown the assertion for the case DS. For the cases DD, SS the proof is similar, and omitted. \square

7 The eigenspaces of X

Let \mathcal{V} denote an XD \hat{H}_q -module. In this section we show that X is multiplicity-free on \mathcal{V} . We then introduce the notion of an X -standard basis. Consider the X -diagram of \mathcal{V} , which we are not assuming is reduced. We use the following term.

Definition 7.1 Let μ, ν denote distinct vertices of the X -diagram that are 1-adjacent or q -adjacent. For $u \in \mathcal{V}_X(\mu)$ and $v \in \mathcal{V}_X(\nu)$, we say that v *follows* u whenever $v = Gu$, where $G = G_0$ if μ, ν are 1-adjacent and $G = G_2$ if μ, ν are q -adjacent.

For the rest of this section, we fix a standard ordering $\{\mu_r\}_{r=0}^n$ of the eigenvalues of X .

Definition 7.2 By a *forward chain* in \mathcal{V} (with respect to the ordering $\{\mu_r\}_{r=0}^n$) we mean a sequence of vectors $\{v_r\}_{r=0}^n$ in \mathcal{V} such that

- (i) $v_r \in \mathcal{V}_X(\mu_r)$ for $0 \leq r \leq n$;
- (ii) $v_0 \neq 0$;
- (iii) v_r follows v_{r-1} for $1 \leq r \leq n$.

Definition 7.3 By a *backward chain* in \mathcal{V} (with respect to the ordering $\{\mu_r\}_{r=0}^n$) we mean a sequence of vectors $\{v_r\}_{r=0}^n$ in \mathcal{V} such that

- (i) $v_r \in \mathcal{V}_X(\mu_r)$ for $0 \leq r \leq n$;
- (ii) $v_n \neq 0$;
- (iii) v_{r-1} follows v_r for $1 \leq r \leq n$.

Definition 7.4 A sequence of vectors $\{v_r\}_{r=0}^n$ in \mathcal{V} is called a *chain* whenever it is a forward or backward chain.

Lemma 7.5 Assume that the reduced X -diagram of \mathcal{V} is of type DD or SS. Let $\{v_r\}_{r=0}^n$ denote a forward chain (resp. backward chain) with respect to the ordering $\{\mu_r\}_{r=0}^n$. Then $\{v_{n-r}\}_{r=0}^n$ is a backward chain (resp. forward chain) with respect to the ordering $\{\mu_{n-r}\}_{r=0}^n$.

Lemma 7.6 The following hold.

- (i) For a nonzero $v \in \mathcal{V}_X(\mu_0)$, there exists a unique forward chain $\{v_r\}_{r=0}^n$ such that $v_0 = v$.
- (ii) For a nonzero $v \in \mathcal{V}_X(\mu_n)$, there exists a unique backward chain $\{v_r\}_{r=0}^n$ such that $v_n = v$.

Proof. Routine using Lemma 5.1. □

Lemma 7.7 *Let μ denote an endvertex of the X -diagram.*

- (i) *Assume that μ is incident to a double bond in the reduced X -diagram of \mathcal{V} . Then $\mathcal{V}_X(\mu)$ is invariant under t_0 .*
- (ii) *Assume that μ is incident to a single bond in the reduced X -diagram of \mathcal{V} . Then $\mathcal{V}_X(\mu)$ is invariant under t_2 .*

Proof. (i): If μ is incident to a single loop, then $\mu^{-1} = \mu$. If μ is not incident to a single loop, then $\mathcal{V}_X(\mu^{-1}) = 0$. In either case $\mathcal{V}_X(\mu^{-1}) \subseteq \mathcal{V}_X(\mu)$. By Lemma 6.2(i) $\mathcal{V}_X(\mu) + \mathcal{V}_X(\mu^{-1})$ is invariant under t_0 . By these comments $\mathcal{V}_X(\mu)$ is invariant under t_0 .

(ii): Similar. \square

Proposition 7.8 *The following hold.*

- (i) *X is multiplicity-free on \mathcal{V} .*
- (ii) *Every chain in \mathcal{V} forms a basis for \mathcal{V} .*

Proof. We assume that the X -diagram of \mathcal{V} is of type DS; the proof is similar for the other types. Note that $\mathcal{V}_X(\mu_0)$ (resp. $\mathcal{V}_X(\mu_n)$) is invariant under t_0 (resp. t_2) by Lemma 7.7. We claim that a chain $\{v_r\}_{r=0}^n$ in \mathcal{V} forms a basis for \mathcal{V} provided that the following conditions hold:

$$\mathbb{F}v_0 \text{ is invariant under } t_0; \tag{40}$$

$$\mathbb{F}v_n \text{ is invariant under } t_2. \tag{41}$$

Let \mathcal{W} denote the subspace of \mathcal{V} spanned by $\{v_r\}_{r=0}^n$. Note that $\mathcal{W} \neq 0$ by the definition of a chain. Using Lemma 6.4 and the conditions (40), (41) one finds that \mathcal{W} is invariant under each of t_0, t_2 . By this and Lemma 6.1(iii) \mathcal{W} is an \hat{H}_q -submodule of \mathcal{V} , and this forces $\mathcal{W} = \mathcal{V}$ since \mathcal{V} is irreducible. Therefore $\{v_r\}_{r=0}^n$ forms a basis for \mathcal{V} .

(i): We construct a chain $\{v_r\}_{r=0}^n$ that satisfies (40) and (41). By Lemma 6.8 either μ_0 is not incident to a single loop or μ_n is not incident to a double loop. First assume that μ_n is not incident to a double loop. There exists an eigenvector v of t_0 that is contained in $\mathcal{V}_X(\mu_0)$ since $\mathcal{V}_X(\mu_0)$ is invariant under t_0 . By Lemma 7.6(i) there exists a forward chain $\{v_r\}_{r=0}^n$ such that $v_0 = v$. This chain satisfies (40) by the construction, and satisfies (41) by Lemma 6.5(ii). Next assume that μ_0 is not incident to a single loop. We can construct a backward chain $\{v_r\}_{r=0}^n$ that satisfies (40) and (41) in a similar way as above. We have constructed a chain $\{v_r\}_{r=0}^n$ that satisfies (40) and (41). By the claim this chain forms a basis for \mathcal{V} , and so $\mathcal{V}_X(\mu_r)$ is spanned by v_r for $0 \leq r \leq n$. Thus X is multiplicity-free on \mathcal{V} .

(ii): Let $\{v_r\}_{r=0}^n$ denote a chain in \mathcal{V} . This chain satisfies (40) and (41) by (i) and since $\mathcal{V}_X(\mu_0)$ (resp. $\mathcal{V}_X(\mu_n)$) is invariant under t_0 (resp. t_2). By the claim this chain forms a basis for \mathcal{V} . \square

Lemma 7.9 *Let μ, ν denote distinct eigenvalues of X .*

- (i) *Assume that μ, ν are 1-adjacent. Then $G_0\mathcal{V}_X(\mu) = \mathcal{V}_X(\nu)$ and $G_0\mathcal{V}_X(\nu) = \mathcal{V}_X(\mu)$.*
- (ii) *Assume that μ, ν are q -adjacent. Then $G_2\mathcal{V}_X(\mu) = \mathcal{V}_X(\nu)$ and $G_2\mathcal{V}_X(\nu) = \mathcal{V}_X(\mu)$.*

Proof. Routine using Lemma 5.1 and Proposition 7.8(ii). \square

Definition 7.10 By an X -standard basis for \mathcal{V} (corresponding to the ordering $\{\mu_r\}_{r=0}^n$) we mean a basis $\{v_r\}_{r=0}^n$ for \mathcal{V} that is a forward chain.

We mention two corollaries for later use. Let $\{k_i\}_{i \in \mathbb{I}}$ denote a parameter sequence of \mathcal{V} .

Corollary 7.11 *Let μ, ν denote distinct eigenvalues of X .*

- (i) *Assume that μ, ν are 1-adjacent. Then $G(\mu, k_0, k_3) \neq 0$.*
- (ii) *Assume that μ, ν are q -adjacent. Then $G(q\mu, k_1, k_2) \neq 0$.*

Proof. Follows from Corollary 5.4 and Lemma 7.9. \square

Corollary 7.12 *Let μ, ν denote distinct eigenvalues of X .*

- (i) *Assume that μ, ν are 1-adjacent. Then μ is not among*

$$k_0k_3, \quad k_0k_3^{-1}, \quad k_0^{-1}k_3, \quad k_0^{-1}k_3^{-1}.$$

- (ii) *Assume that μ, ν are q -adjacent. Then $q\mu$ is not among*

$$k_1k_2, \quad k_1k_2^{-1}, \quad k_1^{-1}k_2, \quad k_1^{-1}k_2^{-1}.$$

Proof. Follows from Corollary 7.11 and (31). \square

8 The X -type of an XD \hat{H}_q -module

In Section 1 we discussed the five cases DS, DDa, DDb, SSa, SSb with reference to some lemmas. In this section we prove these lemmas. We then define the X -type of an XD \hat{H}_q -module, and make a rule concerning the corresponding parameter sequence $\{k_i\}_{i \in \mathbb{I}}$. Let \mathcal{V} denote an XD \hat{H}_q -module and let $\{\mu_r\}_{r=0}^n$ denote a standard ordering of the eigenvalues of X .

Lemma 8.1 *The subspaces $\mathcal{V}_X(\mu_0)$ and $\mathcal{V}_X(\mu_n)$ are invariant under the following elements:*

X -diagram	$\mathcal{V}_X(\mu_0)$ is invariant under	$\mathcal{V}_X(\mu_n)$ is invariant under
DS	$t_0, \quad t_3$	$t_1, \quad t_2$
DD	$t_0, \quad t_3$	$t_0, \quad t_3$
SS	$t_1, \quad t_2$	$t_1, \quad t_2$

Proof. Follows from Lemmas 6.1 and 7.7. \square

Lemma 8.2 *Assume that the reduced X -diagram of \mathcal{V} is DD. Then one of the following cases occurs:*

Case	Eigenvalues of t_0 on $\mathcal{V}_X(\mu_0)$ and $\mathcal{V}_X(\mu_n)$	Eigenvalues of t_3 on $\mathcal{V}_X(\mu_0)$ and $\mathcal{V}_X(\mu_n)$
DDa	same	reciprocals
DDb	reciprocals	same

Proof. We consider the subspaces $\mathcal{V}_X(\mu_0)$ and $\mathcal{V}_X(\mu_n)$. First assume that the eigenvalues of t_0 on the subspaces are the same. Then the eigenvalues of t_3 are reciprocals; otherwise the eigenvalues of $X = t_3 t_0$ are the same, and this forces $\mu_0 = \mu_n$. Next assume that the eigenvalues of t_0 are reciprocals. Then the eigenvalues of t_3 are the same; otherwise the eigenvalues of X are reciprocals and this forces $q^{-n-1} = 1$ since $\mu_0 \mu_n = q^{-n-1}$ by (39). Thus at least one of the cases DDa, DDb occurs. These cases do not occur at the same time; otherwise the eigenvalues of t_0, t_3 on the subspaces are contained in $\{1, -1\}$ and this forces $q^{-n-1} = \pm 1$ since $\mu_0 \mu_n = q^{-n-1}$ by (39). \square

Lemma 8.3 *Assume that the reduced X -diagram of \mathcal{V} is SS. Then one of the following cases occurs:*

Case	Eigenvalues of t_1 on $\mathcal{V}_X(\mu_0)$ and $\mathcal{V}_X(\mu_n)$	Eigenvalues of t_2 on $\mathcal{V}_X(\mu_0)$ and $\mathcal{V}_X(\mu_n)$
SSa	same	reciprocals
SSb	reciprocals	same

Proof. Similar to the proof of Lemma 8.2 using $X = q^{-1} t_2^{-1} t_1^{-1}$. \square

In view of Lemmas 8.2 and 8.3 we make some definitions.

Definition 8.4 We define the X -type of \mathcal{V} as follows. We say that \mathcal{V} has X -type DS whenever the reduced X -diagram of \mathcal{V} is DS. We say that \mathcal{V} has X -type DDa (resp. DDb) whenever the reduced X -diagram of \mathcal{V} is DD and case DDa (resp. DDb) occurs in Lemma 8.2. We say that \mathcal{V} has X -type SSa (resp. SSb) whenever the reduced X -diagram of \mathcal{V} is SS and case SSa (resp. SSb) occurs in Lemma 8.3.

Definition 8.5 A parameter sequence $\{k_i\}_{i \in \mathbb{I}}$ of \mathcal{V} is said to be *consistent with the ordering* $\{\mu_r\}_{r=0}^n$ whenever it follows the rule:

X-type of \mathcal{V}	Rule
DS	k_0 (resp. k_3) is the eigenvalue of t_0 (resp. t_3) on $\mathcal{V}_X(\mu_0)$ k_1 (resp. k_2) is the eigenvalue of t_1 (resp. t_2) on $\mathcal{V}_X(\mu_n)$
DDa, DDb	k_0 (resp. k_3) is the eigenvalue of t_0 (resp. t_3) on $\mathcal{V}_X(\mu_0)$
SSa, SSb	k_1 (resp. k_2) is the eigenvalue of t_1 (resp. t_2) on $\mathcal{V}_X(\mu_0)$

(42)

Lemma 8.6 Assume that the parameter sequence $\{k_i\}_{i \in \mathbb{I}}$ is consistent with the ordering $\{\mu_r\}_{r=0}^n$. Pick nonzero vectors $v_0 \in \mathcal{V}_X(\mu_0)$ and $v_n \in \mathcal{V}_X(\mu_n)$. Then

X-type of \mathcal{V}	Action
DS	$t_0 v_0 = k_0 v_0$ $t_3 v_0 = k_3 v_0$ $t_1 v_n = k_1 v_n$ $t_2 v_n = k_2 v_n$
DDa	$t_0 v_0 = k_0 v_0$ $t_3 v_0 = k_3 v_0$ $t_0 v_n = k_0 v_n$ $t_3 v_n = k_3^{-1} v_n$
DDb	$t_0 v_0 = k_0 v_0$ $t_3 v_0 = k_3 v_0$ $t_0 v_n = k_0^{-1} v_n$ $t_3 v_n = k_3 v_n$
SSa	$t_1 v_0 = k_1 v_0$ $t_2 v_0 = k_2 v_0$ $t_1 v_n = k_1 v_n$ $t_2 v_n = k_2^{-1} v_n$
SSb	$t_1 v_0 = k_1 v_0$ $t_2 v_0 = k_2 v_0$ $t_1 v_n = k_1^{-1} v_n$ $t_2 v_n = k_2 v_n$

(43)

Proof. Immediate from Definitions 8.4 and 8.5. □

By the construction, among the parameter sequences $\{k_i^{\pm 1}\}_{i \in \mathbb{I}}$ of \mathcal{V} , at least one sequence is consistent with the ordering $\{\mu_r\}_{r=0}^n$. Fix a parameter sequence $\{k_i\}_{i \in \mathbb{I}}$ of \mathcal{V} that is consistent with the ordering $\{\mu_r\}_{r=0}^n$. Then the parameter sequences of \mathcal{V} that are consistent with the ordering $\{\mu_r\}_{r=0}^n$ are as follows:

X-type of \mathcal{V}	Parameter sequences
DS	(k_0, k_1, k_2, k_3)
DDa, DDb	$(k_0, k_1^{\pm 1}, k_2^{\pm 1}, k_3)$
SSa, SSb	$(k_0^{\pm 1}, k_1, k_2, k_3^{\pm 1})$

Assume for the moment that \mathcal{V} has X-type among DDa, DDb, SSa, SSb. Recall that the ordering $\{\mu_{n-r}\}_{r=0}^n$ is also standard. The parameter sequences of \mathcal{V} that are consistent with the ordering $\{\mu_{n-r}\}_{r=0}^n$ are as follows:

X-type of \mathcal{V}	Parameter sequences
DDa	$(k_0, k_1^{\pm 1}, k_2^{\pm 1}, k_3^{-1})$
DDb	$(k_0^{-1}, k_1^{\pm 1}, k_2^{\pm 1}, k_3)$
SSa	$(k_0^{\pm 1}, k_1, k_2^{-1}, k_3^{\pm 1})$
SSb	$(k_0^{\pm 1}, k_1^{-1}, k_2, k_3^{\pm 1})$

By the above observations, the parameter sequences of \mathcal{V} that are consistent with a standard ordering of the eigenvalues of X are as follows:

X -type of \mathcal{V}	Parameter sequences
DS	(k_0, k_1, k_2, k_3)
DDa	$(k_0, k_1^{\pm 1}, k_2^{\pm 1}, k_3^{\pm 1})$
DDb	$(k_0^{\pm 1}, k_1^{\pm 1}, k_2^{\pm 1}, k_3)$
SSa	$(k_0^{\pm 1}, k_1, k_2^{\pm 1}, k_3^{\pm 1})$
SSb	$(k_0^{\pm 1}, k_1^{\pm 1}, k_2, k_3^{\pm 1})$

Note by this that a parameter sequence of \mathcal{V} may be consistent with no standard ordering of the eigenvalues of X .

We mention two lemmas for later use.

Lemma 8.7 *Let $\{k_i\}_{i \in \mathbb{I}}$ denote a parameter sequence of \mathcal{V} that is consistent with the ordering $\{\mu_r\}_{r=0}^n$.*

(i) *Assume that \mathcal{V} has X -type among DS, DDa, DDb. Then*

$$\mu_r = \begin{cases} k_0 k_3 q^r & \text{if } r \text{ is even,} \\ \frac{1}{k_0 k_3 q^{r+1}} & \text{if } r \text{ is odd} \end{cases} \quad (0 \leq r \leq n). \quad (44)$$

(ii) *Assume that \mathcal{V} has X -type among SSa, SSb. Then*

$$\mu_r = \begin{cases} \frac{1}{k_1 k_2 q^{r+1}} & \text{if } r \text{ is even,} \\ k_1 k_2 q^r & \text{if } r \text{ is odd} \end{cases} \quad (0 \leq r \leq n). \quad (45)$$

Proof. (i): By (42) t_0 (resp. t_3) has eigenvalue k_0 (resp. k_3) on $\mathcal{V}_X(\mu_0)$. By this and $X = t_3 t_0$ we get $\mu_0 = k_0 k_3$. Now (44) follows from (38).

(ii): Similar to the proof of (i) using $X = q^{-1} t_2^{-1} t_1^{-1}$. \square

Lemma 8.8 *In the reduced X -diagram of \mathcal{V} , consider an endvertex μ .*

(i) *Assume that μ is not incident to a double bond. Then $G_2 \mathcal{V}_X(\mu) = 0$.*

(ii) *Assume that μ is not incident to a single bond. Then $G_0 \mathcal{V}_X(\mu) = 0$.*

Proof. (i): Pick a nonzero $v \in \mathcal{V}_X(\mu)$. Note that $\mathcal{V}_X(\mu) = \mathbb{F}v$. By Lemma 8.1 $\mathcal{V}_X(\mu)$ is invariant under t_1, t_2 . So v is an eigenvector of t_1 and t_2 . By these comments t_1 and t_2 commute on $\mathcal{V}_X(\mu)$. By this and Definition 4.1 we get $G_2 \mathcal{V}_X(\mu) = 0$.

(ii): Similar. \square

9 The action of t_i on the X -standard basis

Let \mathcal{V} denote an XD \hat{H}_q -module that has parameter sequence $\{k_i\}_{i \in \mathbb{I}}$. Let $\{\mu_r\}_{r=0}^n$ denote a standard ordering of the eigenvalues of X , and let $\{v_r\}_{r=0}^n$ denote a corresponding X -standard basis for \mathcal{V} . In this section we display the actions of $\{t_i\}_{i \in \mathbb{I}}$ on $\{v_r\}_{r=0}^n$.

Lemma 9.1 *For $0 \leq r \leq n-1$ such that μ_r, μ_{r+1} are 1-adjacent, the elements t_0, t_3 act on $\mathbb{F}v_r + \mathbb{F}v_{r+1}$ as*

$$t_0 v_r = \frac{\mu_r(k_0 + k_0^{-1}) - k_3 - k_3^{-1}}{\mu_r - \mu_r^{-1}} v_r + \frac{\mu_r}{\mu_r - \mu_r^{-1}} v_{r+1}, \quad (46)$$

$$t_0 v_{r+1} = \frac{G(\mu_r, k_0, k_3)}{\mu_r(\mu_r^{-1} - \mu_r)} v_r + \frac{\mu_r^{-1}(k_0 + k_0^{-1}) - k_3 - k_3^{-1}}{\mu_r^{-1} - \mu_r} v_{r+1}, \quad (47)$$

$$t_3 v_r = \frac{\mu_r(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\mu_r - \mu_r^{-1}} v_r + \frac{1}{\mu_r^{-1} - \mu_r} v_{r+1}, \quad (48)$$

$$t_3 v_{r+1} = \frac{G(\mu_r, k_0, k_3)}{\mu_r - \mu_r^{-1}} v_r + \frac{\mu_r^{-1}(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\mu_r^{-1} - \mu_r} v_{r+1}. \quad (49)$$

In (46)–(49) the denominators are nonzero by Lemma 5.5.

Proof. Recall that for $i \in \mathbb{I}$ the element T_i acts on \mathcal{V} as the scalar $k_i + k_i^{-1}$. Note that $v_{r+1} = G_0 v_r$ by the construction. Now applying (33) to v_r one finds (46). To get (47), apply (33) to $G_0 v_r$ and use Corollary 5.4(i). The lines (48) and (49) are similarly obtained from (34). \square

Lemma 9.2 *For $0 \leq r \leq n-1$ such that μ_r, μ_{r+1} are q -adjacent, the elements t_1, t_2 act on $\mathbb{F}v_r + \mathbb{F}v_{r+1}$ as*

$$t_1 v_r = \frac{q^{-1}\mu_r^{-1}(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\mu_r^{-1} - q\mu_r} v_r + \frac{1}{q\mu_r - q^{-1}\mu_r^{-1}} v_{r+1}, \quad (50)$$

$$t_1 v_{r+1} = \frac{G(q\mu_r, k_1, k_2)}{q^{-1}\mu_r^{-1} - q\mu_r} v_r + \frac{q\mu_r(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{q\mu_r - q^{-1}\mu_r^{-1}} v_{r+1}, \quad (51)$$

$$t_2 v_r = \frac{q^{-1}\mu_r^{-1}(k_2 + k_2^{-1}) - k_1 - k_1^{-1}}{q^{-1}\mu_r^{-1} - q\mu_r} v_r + \frac{q^{-1}\mu_r^{-1}}{q^{-1}\mu_r^{-1} - q\mu_r} v_{r+1}, \quad (52)$$

$$t_2 v_{r+1} = \frac{q\mu_r G(q\mu_r, k_1, k_2)}{q\mu_r - q^{-1}\mu_r^{-1}} v_r + \frac{q\mu_r(k_2 + k_2^{-1}) - k_1 - k_1^{-1}}{q\mu_r - q^{-1}\mu_r^{-1}} v_{r+1}. \quad (53)$$

In (50)–(53) the denominators are nonzero by Lemma 5.6.

Proof. Similar to the proof of Lemma 9.1. \square

In the above, we displayed some actions of $\{t_i\}_{i \in \mathbb{I}}$ on $\{v_i\}_{i \in \mathbb{I}}$. The remaining actions are given in (43).

Note 9.3 By Lemmas 8.6, 9.1, 9.2 the action of $\{t_i\}_{i \in \mathbb{I}}$ on an X -standard basis is determined by the parameter sequence and the X -type of the \hat{H}_q -module. Therefore an XD \hat{H}_q -module is uniquely determined up to isomorphism by its dimension, its parameter sequence, and its X -type.

10 The eigenspaces of t_0

Throughout this section the following notation is in effect.

Notation 10.1 Let \mathcal{V} denote an XD \hat{H}_q -module with dimension $n+1$. Let $\{\mu_r\}_{r=0}^n$ denote a standard ordering of the eigenvalues of X . Let $\{k_i\}_{i \in \mathbb{I}}$ denote a parameter sequence of \mathcal{V} that is consistent with the ordering $\{\mu_r\}_{r=0}^n$.

In this section we obtain the dimensions of the eigenspaces of t_0 . Define subspaces of \mathcal{V} :

$$\mathcal{V}(k_0) = \{v \in \mathcal{V} \mid t_0 v = k_0 v\}, \quad \mathcal{V}(k_0^{-1}) = \{v \in \mathcal{V} \mid t_0 v = k_0^{-1} v\}. \quad (54)$$

Thus $\mathcal{V}(k_0)$ is the eigenspace of t_0 associated with eigenvalue k_0 , provided that $\mathcal{V}(k_0) \neq 0$. Similar for $\mathcal{V}(k_0^{-1})$.

Definition 10.2 Assume $k_0 \neq k_0^{-1}$. Define elements of \hat{H}_q :

$$F^+ = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}, \quad F^- = \frac{t_0 - k_0}{k_0^{-1} - k_0}. \quad (55)$$

Assume for the moment that $k_0 \neq k_0^{-1}$. Recall that $(t_0 - k_0)(t_0 - k_0^{-1})\mathcal{V} = 0$. So

$$\mathcal{V} = \mathcal{V}(k_0) + \mathcal{V}(k_0^{-1}) \quad (\text{direct sum}).$$

Observe that F^+ (resp. F^-) acts on \mathcal{V} as the projection onto $\mathcal{V}(k_0)$ (resp. $\mathcal{V}(k_0^{-1})$).

Lemma 10.3 Let μ, ν denote a single bond in the reduced X -diagram of \mathcal{V} . Then

$$t_0 \mathcal{V}_X(\mu) \subseteq \mathcal{V}_X(\mu) + \mathcal{V}_X(\nu), \quad t_0 \mathcal{V}_X(\mu) \not\subseteq \mathcal{V}_X(\mu), \quad (56)$$

$$t_0^{-1} \mathcal{V}_X(\mu) \subseteq \mathcal{V}_X(\mu) + \mathcal{V}_X(\nu), \quad t_0^{-1} \mathcal{V}_X(\mu) \not\subseteq \mathcal{V}_X(\mu). \quad (57)$$

Proof. By Lemma 7.9(i) $G_0 \mathcal{V}_X(\mu) = \mathcal{V}_X(\nu)$. The result follows from this and Lemma 5.5. \square

Lemma 10.4 Assume $k_0 \neq k_0^{-1}$. Let μ, ν denote a single bond in the reduced X -diagram of \mathcal{V} . Then for $0 \neq v \in \mathcal{V}_X(\mu)$ each of $F^+ v$ and $F^- v$ is nonzero and contained in $\mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$.

Proof. By Lemma 10.3 there exist $\alpha \in \mathbb{F}$ and $0 \neq u \in \mathcal{V}_X(\nu)$ such that $t_0 v = \alpha v + u$. Using this and (55) we argue

$$(k_0 - k_0^{-1})F^+ v = t_0 v - k_0^{-1} v = (\alpha - k_0^{-1})v + u.$$

Thus F^+v is nonzero and contained in $\mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$. The proof is similar for F^-v . \square

Lemma 10.5 *Assume $k_0 \neq k_0^{-1}$. Let μ, ν denote a single bond in the reduced X -diagram of \mathcal{V} , and set $\mathcal{W} = \mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$. Then the following hold:*

- (i) $\mathcal{W} = F^+\mathcal{W} + F^-\mathcal{W}$ (direct sum);
- (ii) each of $F^+\mathcal{W}$ and $F^-\mathcal{W}$ has dimension 1;
- (iii) $F^+\mathcal{W} = \mathcal{W} \cap \mathcal{V}(k_0)$ and $F^-\mathcal{W} = \mathcal{W} \cap \mathcal{V}(k_0^{-1})$;
- (iv) each of $F^+\mathcal{V}_X(\mu)$ and $F^+\mathcal{V}_X(\nu)$ is equal to $F^+\mathcal{W}$;
- (v) each of $F^-\mathcal{V}_X(\mu)$ and $F^-\mathcal{V}_X(\nu)$ is equal to $F^-\mathcal{W}$.

Proof. By Lemma 10.4 each of $F^+\mathcal{V}_X(\mu)$ and $F^-\mathcal{V}_X(\nu)$ is nonzero and contained in \mathcal{W} . Similarly, each of $F^-\mathcal{V}_X(\mu)$ and $F^+\mathcal{V}_X(\nu)$ is nonzero and contained in \mathcal{W} . By these comments, each of $F^+\mathcal{W}$ and $F^-\mathcal{W}$ is nonzero. By the construction $\dim \mathcal{W} = 2$ and $F^+\mathcal{W} \cap F^-\mathcal{W} = 0$. Now (i)–(v) follows from these comments. \square

Lemma 10.6 *Assume $k_0 \neq k_0^{-1}$. Consider the reduced X -diagram of \mathcal{V} . Then the dimension of $\mathcal{V}(k_0)$ (resp. $\mathcal{V}(k_0^{-1})$) is equal to the number of single bonds plus the number of endvertices μ such that μ is incident to a double bond and $\mathcal{V}_X(\mu)$ is contained in $\mathcal{V}(k_0)$ (resp. $\mathcal{V}(k_0^{-1})$).*

Proof. Follows from Lemmas 8.1 and 10.5. \square

Lemma 10.7 *The following hold.*

- (i) Assume that \mathcal{V} has X -type among DS, DDa, DDb. Then $\mathcal{V}_X(\mu_0)$ is contained in $\mathcal{V}(k_0)$.
- (ii) Assume that \mathcal{V} has X -type DDa. Then $\mathcal{V}_X(\mu_n)$ is contained in $\mathcal{V}(k_0)$.
- (iii) Assume that \mathcal{V} has X -type DDb. Then $\mathcal{V}_X(\mu_n)$ is contained in $\mathcal{V}(k_0^{-1})$.

Proof. Use Lemma 8.6. \square

Lemma 10.8 *Assume $n \geq 1$ and $k_0 \neq k_0^{-1}$. Then the following are equivalent:*

- (i) t_0 has only one eigenvalue on \mathcal{V} ;
- (ii) $n = 1$ and \mathcal{V} has X -type DDa.

Proof. (i) \Rightarrow (ii): Assume that t_0 has only one eigenvalue on \mathcal{V} . Then the reduced X -diagram contains no single bond; otherwise each of $\mathcal{V}(k_0^{\pm 1})$ is nonzero by Lemma 10.5. So the reduced X -diagram is DD and $n = 1$. By this and Lemma 10.7 \mathcal{V} has X -type DDa.

(ii) \Rightarrow (i): Assume that $n = 1$ and \mathcal{V} has X -type DDa. Note that $\mathcal{V} = \mathcal{V}_X(\mu_0) + \mathcal{V}_X(\mu_1)$. By Lemma 10.7(i), (ii) each of $\mathcal{V}_X(\mu_0)$, $\mathcal{V}_X(\mu_1)$ is contained in $\mathcal{V}(k_0)$. Thus t_0 has only one eigenvalue k_0 . \square

For the rest of this section the following notation is in effect.

Notation 10.9 Assume that t_0 has two distinct eigenvalues on \mathcal{V} . Define $d = \dim \mathcal{V}(k_0) - 1$ and $d' = \dim \mathcal{V}(k_0^{-1}) - 1$.

Lemma 10.10 *The $d - d'$ is as follows:*

X -type of \mathcal{V}	$d - d'$
DS	1
DDa	2
DDb, SSa, SSb	0

Proof. Follows from Lemmas 10.6 and 10.7. \square

Corollary 10.11 *In the table below we express d and d' in terms of n :*

X -type of \mathcal{V}	d	d'
DS	$n/2$	$(n - 2)/2$
DDa	$(n + 1)/2$	$(n - 3)/2$
DDb	$(n - 1)/2$	$(n - 1)/2$
SSa, SSb	$(n - 1)/2$	$(n - 1)/2$

Proof. Follows from Lemma 10.10. \square

11 Proof of Theorem 1.7

In this section we prove Theorem 1.7. Until above Lemma 11.13, Notation 10.1 is in effect. Referring to Notation 10.9, for each $\mathcal{V}(k_0^{\pm 1})$ we construct an eigenbasis of \mathbb{B} on which \mathbb{A} acts in an irreducible tridiagonal manner.

Lemma 11.1 *With reference to Notation 10.9, there exist nonzero vectors $\{w_r\}_{r=0}^d$ in $\mathcal{V}(k_0)$ that satisfy the following conditions:*

X -type of \mathcal{V}	Conditions	
DS	$w_0 \in \mathcal{V}_X(\mu_0)$ $w_r \in \mathcal{V}_X(\mu_{2r-1}) + \mathcal{V}_X(\mu_{2r}) \quad (1 \leq r \leq d)$	
DDa	$w_0 \in \mathcal{V}_X(\mu_0)$ $w_r \in \mathcal{V}_X(\mu_{2r-1}) + \mathcal{V}_X(\mu_{2r}) \quad (1 \leq r \leq d-1)$ $w_d \in \mathcal{V}_X(\mu_n)$	(58)
DDb	$w_0 \in \mathcal{V}_X(\mu_0)$ $w_r \in \mathcal{V}_X(\mu_{2r-1}) + \mathcal{V}_X(\mu_{2r}) \quad (1 \leq r \leq d)$	
SSa, SSb	$w_r \in \mathcal{V}_X(\mu_{2r}) + \mathcal{V}_X(\mu_{2r+1}) \quad (0 \leq r \leq d)$	

Proof. Consider the reduced X -diagram of \mathcal{V} . First assume that \mathcal{V} has X -type DS. Clearly there exists a nonzero w_0 in $\mathcal{V}_X(\mu_0)$. By Lemma 10.7(i) w_0 is contained in $\mathcal{V}(k_0)$. Pick any r ($1 \leq r \leq d$). Observe that μ_{2r-1}, μ_{2r} is a single bond by the shape of the X -diagram. Define $\mathcal{W} = \mathcal{V}_X(\mu_{2r-1}) + \mathcal{V}_X(\mu_{2r})$. By Lemma 10.5(ii), (iii) $\mathcal{W} \cap \mathcal{V}(k_0)$ has dimension one. Thus there exists a nonzero $w_r \in \mathcal{V}(k_0)$ that is contained in $\mathcal{V}_X(\mu_{2r-1}) + \mathcal{V}_X(\mu_{2r})$. We have shown the result for the case DS. The proof is similar for the other cases. \square

Lemma 11.2 *With reference to Notation 10.9, there exist nonzero vectors $\{w'_r\}_{r=0}^d$ in $\mathcal{V}(k_0^{-1})$ that satisfy the following conditions:*

X -type of \mathcal{V}	Conditions	
DS	$w'_r \in \mathcal{V}_X(\mu_{2r+1}) + \mathcal{V}_X(\mu_{2r+2}) \quad (0 \leq r \leq d')$	
DDa	$w'_r \in \mathcal{V}_X(\mu_{2r+1}) + \mathcal{V}_X(\mu_{2r+2}) \quad (0 \leq r \leq d')$	
DDb	$w'_r \in \mathcal{V}_X(\mu_{2r+1}) + \mathcal{V}_X(\mu_{2r+2}) \quad (0 \leq r \leq d'-1)$ $w'_{d'} \in \mathcal{V}_X(\mu_n)$	(59)
SSa, SSb	$w'_r \in \mathcal{V}_X(\mu_{2r}) + \mathcal{V}_X(\mu_{2r+1}) \quad (0 \leq r \leq d')$	

Proof. Similar to the proof of Lemma 11.1. \square

Lemma 11.3 *With reference to Lemmas 11.1 and 11.2, the vectors $\{w_r\}_{r=0}^d$ (resp. $\{w'_r\}_{r=0}^{d'}$) form a basis for $\mathcal{V}(k_0)$ (resp. $\mathcal{V}(k_0^{-1})$).*

Proof. First assume that \mathcal{V} has X -type DS. To simplify notation, set $\mathcal{W}_0 = \mathcal{V}_X(\mu_0)$ and $\mathcal{W}_r = \mathcal{V}_X(\mu_{r-1}) + \mathcal{V}_X(\mu_r)$ for $1 \leq r \leq d$. Observe that $\mathcal{V} = \sum_{r=0}^d \mathcal{W}_r$ (direct sum). By the conditions in Lemma 11.1 $w_r \in \mathcal{W}_r \cap \mathcal{V}(k_0)$ for $0 \leq r \leq d$. By Lemma 10.7(i) and Lemma 10.5(iii) $\mathcal{W}_r \cap \mathcal{V}(k_0) = F^+ \mathcal{W}_r$ for $0 \leq r \leq d$. By these comments $w_r \in F^+ \mathcal{W}_r$ for $0 \leq r \leq d$. By this and $F^+ \mathcal{V} = \sum_{r=0}^d F^+ \mathcal{W}_r$ (direct sum), the vectors $\{w_r\}_{r=0}^d$ form a basis

for $F^+\mathcal{V} = \mathcal{V}(k_0)$. Similarly $\{w'_r\}_{r=0}^{d'}$ form a basis for $\mathcal{V}(k_0^{-1})$. We have shown the result for the case DS. The proof is similar for the other cases. \square

Note that each of \mathbb{A} and \mathbb{B} commutes with t_0 by Lemma 3.6(ii), and so each of $\mathcal{V}(k_0)$ and $\mathcal{V}(k_0^{-1})$ is invariant under \mathbb{A}, \mathbb{B} . We first consider the action of \mathbb{B} on $\mathcal{V}(k_0^{\pm 1})$.

Lemma 11.4 *The following hold.*

- (i) *Let μ denote a vertex in the X -diagram of \mathcal{V} . Then $\mathcal{V}_X(\mu)$ is contained in the eigenspace of \mathbb{B} with eigenvalue $\mu + \mu^{-1}$.*
- (ii) *Let μ, ν denote a single bond in the reduced X -diagram of \mathcal{V} . Then $\mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$ is contained in the eigenspace of \mathbb{B} with eigenvalue $\mu + \mu^{-1}$.*

Proof. (i): Clear by $\mathbb{B} = X + X^{-1}$.

(ii): We have $\nu = \mu^{-1}$ since μ, ν are 1-adjacent. So each of $\mathcal{V}_X(\mu)$ and $\mathcal{V}_X(\nu)$ is contained in the eigenspace of \mathbb{B} for the eigenvalue $\mu + \mu^{-1}$. \square

Lemma 11.5 *With reference to Notation 10.9, let the bases $\{w_r\}_{r=0}^d$ and $\{w'_r\}_{r=0}^{d'}$ be from Lemma 11.3. Then the vectors $\{w_r\}_{r=0}^d$ and $\{w'_r\}_{r=0}^{d'}$ are eigenvectors of \mathbb{B} with the following eigenvalues:*

X -type of \mathcal{V}	Eigenvalue of \mathbb{B} for w_r	Eigenvalue of \mathbb{B} for w'_r
DS, DDa, DDb	$k_0 k_3 q^{2r} + \frac{1}{k_0 k_3 q^{2r}}$	$k_0 k_3 q^{2r+2} + \frac{1}{k_0 k_3 q^{2r+2}}$
SSa, SSb	$k_1 k_2 q^{2r+1} + \frac{1}{k_1 k_2 q^{2r+1}}$	$k_1 k_2 q^{2r+1} + \frac{1}{k_1 k_2 q^{2r+1}}$

Proof. Follows from Lemmas 8.7 and 11.4. \square

Next we consider the action of \mathbb{A} on $\mathcal{V}(k_0^{\pm 1})$.

Lemma 11.6 *Assume $k_0 \neq k_0^{-1}$. Then the following hold on \mathcal{V} :*

$$\mathbb{A}F^+ = (k_0 - k_0^{-1})F^+ t_1 F^+ + k_0^{-1}(k_1 + k_1^{-1})F^+, \quad (60)$$

$$\mathbb{A}F^- = (k_0^{-1} - k_0)F^- t_1 F^- + k_0(k_1 + k_1^{-1})F^-. \quad (61)$$

Proof. We first show (60). We have $t_0^{-1}F^+ = k_0^{-1}F^+$ on \mathcal{V} since F^+ acts on \mathcal{V} as the projection onto $\mathcal{V}(k_0)$. By (4) we have $T_1 = k_1 + k_1^{-1}$ on \mathcal{V} . Using these comments and $t_1^{-1} = T_1 - t_1$ we argue on \mathcal{V}

$$\begin{aligned} \mathbb{A}F^+ &= (t_0 t_1 + t_1^{-1} t_0^{-1})F^+ \\ &= t_0 t_1 F^+ + k_0^{-1}(T_1 - t_1)F^+ \\ &= t_0 t_1 F^+ + k_0^{-1}(k_1 + k_1^{-1})F^+ - k_0^{-1}t_1 F^+ \\ &= (t_0 - k_0^{-1})t_1 F^+ + k_0^{-1}(k_1 + k_1^{-1})F^+. \end{aligned}$$

Now (60) follows since $t_0 - k_0^{-1} = (k_0 - k_0^{-1})F^+$. The proof of (61) is similar. \square

Lemma 11.7 Assume $k_0 \neq k_0^{-1}$. Let μ, ν denote a single bond in the reduced X -diagram of \mathcal{V} . Then for $0 \neq v \in \mathcal{V}_X(\mu)$ the following hold.

- (i) There exist nonzero $a, b \in \mathbb{F}$ such that $F^+v = av + bG_0v$.
- (ii) There exist nonzero $a, b \in \mathbb{F}$ such that $F^-v = av + bG_0v$.

Proof. (i): Note that v is a basis for $\mathcal{V}_X(\mu)$. By Lemma 7.9(i) G_0v is a basis for $\mathcal{V}_X(\nu)$. By Lemma 10.4 F^+v is nonzero and contained in $\mathcal{V}_X(\mu) + \mathcal{V}_X(\nu)$. By these comments there exist $a, b \in \mathbb{F}$ such that $F^+v = av + bG_0v$. We show $b \neq 0$. By way of contradiction, assume $b = 0$. Then F^+v is contained in $\mathcal{V}_X(\mu)$, and so F^+v is a basis for $\mathcal{V}_X(\mu)$. This forces $F^-\mathcal{V}_X(\mu) = 0$; this is a contradiction since $F^-v \neq 0$ by Lemma 10.4. Thus $b \neq 0$. We can show $a \neq 0$ in a similar way.

(ii): Similar. □

Lemma 11.8 Let μ denote an eigenvalue of X and let $0 \neq v \in \mathcal{V}_X(\mu)$. Then there exist $e, f \in \mathbb{F}$ with $f \neq 0$ such that

$$t_1v = ev + fG_2v.$$

Proof. We may assume $n \geq 1$; otherwise the result follows since $G_2v = 0$ by Lemma 8.8(i). First assume that μ is not incident to a double bond in the reduced X -diagram of \mathcal{V} . Then μ is an endvertex of the reduced X -diagram of \mathcal{V} that is incident to a single bond. By Lemma 8.1 $\mathcal{V}_X(\mu)$ is invariant under t_1 . By Lemma 8.8(i) $G_2v = 0$. The result follows from these comments. Next assume that μ is incident to a double bond in the reduced X -diagram of \mathcal{V} . Then the result follows from Lemma 5.6. □

Lemma 11.9 Assume $k_0 \neq k_0^{-1}$. Let μ denote an eigenvalue of X and let $0 \neq v \in \mathcal{V}_X(\mu)$.

- (i) Assume $F^+v \neq 0$. Then there exist $\alpha, \beta, \gamma \in \mathbb{F}$ with $\beta\gamma \neq 0$ such that

$$F^+t_1F^+v = \alpha F^+v + \beta F^+G_2v + \gamma F^+G_2G_0v.$$

- (ii) Assume $F^-v \neq 0$. Then there exist $\alpha, \beta, \gamma \in \mathbb{F}$ with $\beta\gamma \neq 0$ such that

$$F^-t_1F^-v = \alpha F^-v + \beta F^-G_2v + \gamma F^-G_2G_0v.$$

Proof. (i): We assume $n \geq 1$; otherwise the assertion holds since $G_0v = 0$ and $G_2v = 0$ by Lemma 8.8. By Lemma 11.8 there exist $e, f \in \mathbb{F}$ with $f \neq 0$ such that

$$t_1v = ev + fG_2v.$$

In this equation, apply F^+ to each side to get

$$F^+t_1v = eF^+v + fF^+G_2v. \tag{62}$$

Similarly, there exist $g, h \in \mathbb{F}$ with $h \neq 0$ such that

$$F^+t_1G_0v = gF^+G_0v + hF^+G_2G_0v. \quad (63)$$

First assume that μ is incident to a single bond in the reduced X -diagram of \mathcal{V} . By Lemma 11.7(i) there exist nonzero $a, b \in \mathbb{F}$ such that

$$F^+v = av + bG_0v.$$

In this equation, apply F^+t_1 to each side to get

$$F^+t_1F^+v = aF^+t_1v + bF^+t_1G_0v. \quad (64)$$

Combining (62)–(64)

$$F^+t_1F^+v = aeF^+v + bgF^+G_0v + afF^+G_2v + bhF^+G_2G_0v.$$

The result follows since $af \neq 0$, $bh \neq 0$, and $F^+G_0v \in \text{Span}\{F^+v\}$ by Lemma 10.5(iv). Next assume that μ is not incident to a single bond in the reduced X -diagram of \mathcal{V} . Then μ is an endvertex that is incident to a double bond. By Lemma 8.8(ii) $G_0v = 0$. By Lemma 8.1 v is an eigenvector for t_0 . So either $t_0v = k_0v$ or $t_0v = k_0^{-1}v$. By this and $F^+v \neq 0$ we get $F^+v = v$. By this and (62),

$$F^+t_1F^+v = eF^+v + fF^+G_2v.$$

The result follows since $G_0v = 0$ and $f \neq 0$.

(ii): Similar. □

Lemma 11.10 *Assume $k_0 \neq k_0^{-1}$. Let μ denote an eigenvalue of X and let $0 \neq v \in \mathcal{V}_X(\mu)$.*

(i) *Assume $F^+v \neq 0$. Then there exist $\alpha, \beta, \gamma \in \mathbb{F}$ with $\beta\gamma \neq 0$ such that*

$$\mathbb{A}F^+v = \alpha F^+v + \beta F^+G_2v + \gamma F^+G_2G_0v.$$

(ii) *Assume $F^-v \neq 0$. Then there exist $\alpha, \beta, \gamma \in \mathbb{F}$ with $\beta\gamma \neq 0$ such that*

$$\mathbb{A}F^-v = \alpha F^-v + \beta F^-G_2v + \gamma F^-G_2G_0v.$$

Proof. Follows from Lemmas 11.6 and 11.9. □

Lemma 11.11 *With reference to Notation 10.9, let the bases $\{w_r\}_{r=0}^d$ and $\{w'_r\}_{r=0}^{d'}$ be from Lemma 11.3. Then with respect to the basis $\{w_r\}_{r=0}^d$ (resp. $\{w'_r\}_{r=0}^{d'}$) the matrix representing \mathbb{A} is irreducible tridiagonal.*

Proof. First assume that \mathcal{V} has X -type DS. Note that $d = n/2$ by Corollary 10.11. We assume $d \geq 1$; otherwise the assertion is obviously true. We abbreviate $\mathcal{V}_r = \mathcal{V}_X(\mu_r)$ for $0 \leq r \leq n$. Define subspaces

$$\mathcal{W}_0 = \mathcal{V}_0, \quad \mathcal{W}_r = \mathcal{V}_{2r-1} + \mathcal{V}_{2r} \quad (1 \leq r \leq d).$$

By the construction, $w_r \in F^+\mathcal{W}_r$ for $0 \leq r \leq d$. First we consider the action of \mathbb{A} on w_0 . By the construction $F^+w_0 = w_0 \neq 0$. By Lemma 8.8(ii) $G_0w_0 = 0$. By these comments and Lemma 11.10(i), there exist $\alpha, \beta \in \mathbb{F}$ such that

$$\mathbb{A}w_0 = \alpha w_0 + \beta F^+G_2w_0, \quad \beta \neq 0. \quad (65)$$

By Lemma 7.9(ii) $G_2\mathcal{V}_0 = \mathcal{V}_1$, and so $0 \neq G_2w_0 \in \mathcal{V}_1$. Applying Lemma 10.5(iv) for $\mu = \mu_1$ and $\nu = \mu_2$, we find that $0 \neq F^+G_2w_0 \in \mathbb{F}w_1$. So there exists a nonzero $\beta' \in \mathbb{F}$ such that $F^+G_2w_0 = \beta'w_1$. By this and (65)

$$\mathbb{A}w_0 = \alpha w_0 + \beta\beta'w_1, \quad \beta\beta' \neq 0. \quad (66)$$

Next we consider the action of \mathbb{A} on w_r for $1 \leq r \leq d-1$. Pick any r such that $1 \leq r \leq d-1$. By Lemma 10.5(iv) there exists $v \in \mathcal{V}_{2r-1}$ such that $w_r = F^+v$. By Lemma 11.10(i) there exist $\alpha, \beta, \gamma \in \mathbb{F}$ such that

$$\mathbb{A}F^+v = \alpha F^+v + \beta F^+G_2v + \gamma F^+G_2G_0v, \quad \beta\gamma \neq 0. \quad (67)$$

By Lemma 7.9

$$G_2\mathcal{V}_{2r-1} = \mathcal{V}_{2r-2}, \quad G_2G_0\mathcal{V}_{2r-1} = \mathcal{V}_{2r+1}.$$

So $0 \neq G_2v \in \mathcal{V}_{2r-2}$ and $0 \neq G_2G_0v \in \mathcal{V}_{2r+1}$. By Lemma 10.5(iv) (or by $F^+\mathcal{W}_0 = \mathcal{V}_0$ if $r = 1$) $F^+\mathcal{W}_{r-1} = F^+\mathcal{V}_{2r-2}$. So $0 \neq F^+G_2v \in F^+\mathcal{W}_{r-1}$. Thus $F^+G_2v = \beta'w_{r-1}$ for some nonzero $\beta' \in \mathbb{F}$. Similarly $F^+\mathcal{W}_{r+1} = F^+\mathcal{V}_{2r+1}$, and so $0 \neq F^+G_2G_0v \in F^+\mathcal{W}_{r+1}$. Thus $F^+G_2G_0v = \gamma'w_{r+1}$ for some nonzero $\gamma' \in \mathbb{F}$. By these comments and (67)

$$\mathbb{A}w_r = \beta\beta'w_{r-1} + \alpha w_r + \gamma\gamma'w_{r+1}, \quad \beta\beta' \neq 0, \quad \gamma\gamma' \neq 0. \quad (68)$$

In a similar way as above, we can show that there exist $\alpha, \beta \in \mathbb{F}$ such that

$$\mathbb{A}w_d = \beta w_{r-1} + \alpha w_r, \quad \beta \neq 0. \quad (69)$$

By (66), (68), (69) we find that the matrix representing \mathbb{A} with respect to $\{w_r\}_{r=0}^d$ is irreducible tridiagonal. In a similar way, we find that the matrix representing \mathbb{A} with respect to $\{w'_r\}_{r=0}^{d'}$ is irreducible tridiagonal. We have shown the result when \mathcal{V} has X -type DS. The proof is similar for the other types. \square

Proposition 11.12 *With reference to Notation 10.9, the following hold.*

- (i) *There exists a basis for $\mathcal{V}(k_0)$ with respect to which the matrix representing \mathbb{A} is irreducible tridiagonal and the matrix representing \mathbb{B} is diagonal with the following (r, r) -entry for $0 \leq r \leq d$:*

X -type of \mathcal{V}	(r, r) -entry for \mathbb{B}
DS, DDa, DDb	$k_0 k_3 q^{2r} + \frac{1}{k_0 k_3 q^{2r}}$
SSa, SSb	$k_1 k_2 q^{2r+1} + \frac{1}{k_1 k_2 q^{2r+1}}$

- (ii) *There exists a basis for $\mathcal{V}(k_0^{-1})$ with respect to which the matrix representing \mathbb{A} is irreducible tridiagonal and the matrix representing \mathbb{B} is diagonal with the following (r, r) -entry for $0 \leq r \leq d'$:*

X -type of \mathcal{V}	(r, r) -entry for \mathbb{B}
DS, DDa, DDb	$k_0 k_3 q^{2r+2} + \frac{1}{k_0 k_3 q^{2r+2}}$
SSa, SSb	$k_1 k_2 q^{2r+1} + \frac{1}{k_1 k_2 q^{2r+1}}$

Proof. Follows from Lemmas 11.5 and 11.11. \square

Lemma 11.13 *Let \mathcal{V} denote a finite dimensional irreducible \hat{H}_q -module with parameter sequence $\{k_i\}_{i \in \mathbb{I}}$. Assume that t_0 has two distinct eigenvalues on \mathcal{V} .*

- (i) *Assume that \mathcal{V} is XD. Then for each of $\mathcal{V}(k_0^{\pm 1})$ there exists a basis with respect to which the matrix representing \mathbb{A} is irreducible tridiagonal and the matrix representing \mathbb{B} is diagonal whose diagonal entries form a q -Racah sequence.*
- (ii) *Assume that \mathcal{V} is YD. Then for each of $\mathcal{V}(k_0^{\pm 1})$ there exists a basis with respect to which the matrix representing \mathbb{B} is irreducible tridiagonal and the matrix representing \mathbb{A} is diagonal whose diagonal entries form a q -Racah sequence.*

Proof. (i): We may assume that $\{k_i\}_{i \in \mathbb{I}}$ is consistent with a standard ordering of the eigenvalues of X on \mathcal{V} , by replacing any of $\{k_i\}_{i \in \mathbb{I}}$ with its inverse if necessary. Now the result follows from Proposition 11.12.

(ii): Recall the automorphism σ of \hat{H}_q from Lemma 3.3. By the definition of σ , t_0 is fixed by σ . By Lemma 3.5 σ sends $Y \mapsto X$ and swaps \mathbb{A}, \mathbb{B} . Consider the \hat{H}_q -module \mathcal{V}^σ , where $\mathcal{V}^\sigma = \mathcal{V}$ as sets, and the action of $x \in \hat{H}_q$ in \mathcal{V}^σ is equal to the action of x^σ in \mathcal{V} . Observe that the \hat{H}_q -module \mathcal{V}^σ is XD. Applying (i) to \mathcal{V}^σ we obtain the result. \square

Proof of Theorem 1.7. Follows from Lemma 11.13. \square

12 The parameter sequence of an XD \hat{H}_q -module

Throughout this section Notation 10.1 is in effect. In this section we investigate the parameter sequence of \mathcal{V} .

Lemma 12.1 *The parameters $\{k_i\}_{i \in \mathbb{I}}$ satisfy the following equation:*

X-type of \mathcal{V}	Equation
DS	$k_0 k_1 k_2 k_3 = q^{-n-1}$
DDa	$k_0^2 = q^{-n-1}$
DDb	$k_3^2 = q^{-n-1}$
SSa	$k_1^2 = q^{-n-1}$
SSb	$k_2^2 = q^{-n-1}$

Proof. Pick nonzero vectors $v_0 \in \mathcal{V}_X(\mu_0)$ and $v_n \in \mathcal{V}_X(\mu_n)$. First assume that \mathcal{V} has X-type DS. By Lemma 8.6 $t_1 v_n = k_1 v_n$ and $t_2 v_n = k_2 v_n$. By this and $X = q^{-1} t_2^{-1} t_1^{-1}$ we obtain $X v_n = q^{-1} k_1^{-1} k_2^{-1} v_n$. So $\mu_n = q^{-1} k_1^{-1} k_2^{-1}$. We have $\mu_n = k_0 k_3 q^n$ by Lemma 8.7(i). Comparing these two values of μ_n , we obtain $k_0 k_1 k_2 k_3 = q^{-n-1}$. Next assume that \mathcal{V} has X-type DDa. By Lemma 8.6 $t_0 v_n = k_0 v_n$ and $t_3 v_n = k_3^{-1} v_n$. So $X v_n = k_0 k_3^{-1} v_n$, and hence $\mu_n = k_0 k_3^{-1}$. By Lemma 8.7(i) $\mu_n = (k_0 k_3 q^{n+1})^{-1}$. Comparing these two values of μ_n , we obtain $k_0^2 = q^{-n-1}$. Next assume that \mathcal{V} has X-type DDb. We obtain $k_3^2 = q^{-n-1}$ in a similar way. Next assume that \mathcal{V} has X-type SSa. By Lemma 8.6 $t_1 v_n = k_1 v_n$ and $t_2 v_n = k_2^{-1} v_n$. By this and $X = q^{-1} t_2^{-1} t_1^{-1}$ we obtain $X v_n = q^{-1} k_1^{-1} k_2 v_n$, and hence $\mu_n = q^{-1} k_1^{-1} k_2$. By Lemma 8.7(ii) $\mu_n = k_1 k_2 q^n$. Comparing these two values of μ_n we obtain $k_1^2 = q^{-n-1}$. Next assume that \mathcal{V} has X-type SSb. We obtain $k_2^2 = q^{-n-1}$ in a similar way. \square

Lemma 12.2 *The parameters $\{k_i\}_{i \in \mathbb{I}}$ satisfy the following inequalities:*

X-type of \mathcal{V}	Inequalities
DS, DDa, DDb	$k_0^2 k_3^2$ is not among $q^{-2}, q^{-4}, q^{-6}, \dots, q^{-2n}$
SSa, SSb	$k_1^2 k_2^2$ is not among $q^{-2}, q^{-4}, q^{-6}, \dots, q^{-2n}$

Proof. First assume that \mathcal{V} has X-type DS. Since $\{\mu_r\}_{r=0}^n$ are mutually distinct, $\mu_{2r} - \mu_{2s-1} \neq 0$ for $0 \leq r \leq n/2$ and $1 \leq s \leq n/2$. By Lemma 8.7(i)

$$\mu_{2r} - \mu_{2s-1} = k_0 k_3 q^{2r} - \frac{1}{k_0 k_3 q^{2s}} = \frac{k_0^2 k_3^2 q^{2(r+s)} - 1}{k_0 k_3 q^{2s}}.$$

By these comments $k_0^2 k_3^2 q^{2(r+s)} \neq 1$ for $0 \leq r \leq n/2$ and $1 \leq s \leq n/2$. So $k_0^2 k_3^2$ is not among $q^{-2}, q^{-4}, \dots, q^{-2n}$. Next assume that \mathcal{V} has X-type DDa or DDb. We have $\mu_{2r} - \mu_{2s-1} \neq 0$ for $0 \leq r \leq (n-1)/2$ and $1 \leq s \leq (n+1)/2$. In a similar way as above, we find that $k_0^2 k_3^2$

is not among $q^{-2}, q^{-4}, \dots, q^{-2n}$. Next assume that \mathcal{V} has X -type SSa or SSb. We have $\mu_{2r} - \mu_{2s-1} \neq 0$ for $0 \leq r \leq (n-1)/2$ and $1 \leq s \leq (n+1)/2$. By Lemma 8.7(ii)

$$\mu_{2r} - \mu_{2s-1} = \frac{1}{k_1 k_2 q^{2r+1}} - k_1 k_2 q^{2s-1} = \frac{1 - k_1^2 k_2^2 q^{2(r+s)}}{k_1 k_2 q^{2r+1}}.$$

By these comments $k_1^2 k_2^2 q^{2(r+s)} \neq 1$ for $0 \leq r \leq (n-1)/2$ and $1 \leq s \leq (n+1)/2$. So $k_0^2 k_3^2$ is not among $q^{-2}, q^{-4}, \dots, q^{-2n}$. \square

Lemma 12.3 *The parameters $\{k_i\}_{i \in \mathbb{I}}$ satisfy the following inequalities:*

X -type of \mathcal{V}	Inequalities
DS	Neither of k_0^2, k_3^2 is among $q^{-2}, q^{-4}, q^{-6}, \dots, q^{-n}$ None of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{1-n}$
DDa, DDb	None of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$
SSa, SSb	None of $k_1 k_2 k_0^{\pm 1} k_3^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$

Proof. First assume that \mathcal{V} has X -type DS. By Corollary 7.12(i) μ_{2r-1} is not among $k_0^{-1} k_3, k_0 k_3^{-1}$ for $1 \leq r \leq n/2$. By Lemma 8.7(i) $\mu_{2r-1} = (k_0 k_3 q^{2r})^{-1}$. By these comments neither of k_0^2, k_3^2 is among $q^{-2}, q^{-4}, \dots, q^{-n}$. By Corollary 7.12(ii) $q\mu_{2r}$ is not among $k_1^{\pm 1} k_2^{\pm 1}$ for $0 \leq r \leq (n-2)/2$. By Lemma 8.7(i) $\mu_{2r} = k_0 k_3 q^{2r}$. By these comments none of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, \dots, q^{1-n}$. Next assume that \mathcal{V} has X -type DDa or DDb. By Corollary 7.12(ii) $q\mu_{2r}$ is not among $k_1^{\pm 1} k_2^{\pm 1}$ for $0 \leq r \leq (n-1)/2$. By Lemma 8.7(i) $\mu_{2r} = k_0 k_3 q^{2r}$. By these comments none of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, \dots, q^{-n}$. Next assume that \mathcal{V} has X -type SSa or SSb. By Corollary 7.12(i) μ_{2r} is not among $k_0^{\pm 1} k_3^{\pm 1}$ for $0 \leq r \leq (n-1)/2$. By Lemma 8.7(ii) $\mu_{2r} = (k_1 k_2 q^{2r+1})^{-1}$. By these comments none of $k_1 k_2 k_0^{\pm 1} k_3^{\pm 1}$ is among $q^{-1}, q^{-3}, \dots, q^{-n}$. \square

Lemma 12.4 *The parameters $\{k_i\}_{i \in \mathbb{I}}$ satisfy the following inequalities:*

X -type of \mathcal{V}	Inequalities
DS	Neither of $\pm k_0 k_3$ is among $q^{-1}, q^{-2}, q^{-3}, \dots, q^{-n}$ None of $\pm k_0, \pm k_1, \pm k_2, \pm k_3$ is among $q^{-1}, q^{-2}, q^{-3}, \dots, q^{-n/2}$
DDa	None of $\pm k_3^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$ None of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$
DDb	None of $\pm k_0^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$ None of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$
SSa	None of $\pm k_2^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$ None of $k_1 k_2 k_0^{\pm 1} k_3^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$
SSb	None of $\pm k_1^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$ None of $k_1 k_2 k_0^{\pm 1} k_3^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$

Proof. First assume that \mathcal{V} has X -type DS. By Lemma 12.2 $k_0^2 k_3^2$ is not among $q^{-2}, q^{-4}, \dots, q^{-2n}$. So neither of $\pm k_0 k_3$ is among $q^{-1}, q^{-2}, \dots, q^{-n}$. By Lemma 12.3 neither of k_0^2, k_3^2

is among $q^{-2}, q^{-4}, \dots, q^{-n}$. So none of $\pm k_0, \pm k_3$ is among $q^{-1}, q^{-2}, \dots, q^{-n/2}$. By Lemma 12.3 $k_0 k_3 k_1^{-1} k_2$ is not among $q^{-1}, q^{-3}, \dots, q^{1-n}$. By Lemma 12.1 $k_0 k_1 k_2 k_3 = q^{-n-1}$. By these comments $k_1^{-2} q^{-n-1}$ is not among $q^{-1}, q^{-3}, \dots, q^{1-n}$. So neither of $\pm k_1$ is among $q^{-1}, q^{-2}, \dots, q^{-n/2}$. Similarly, neither of $\pm k_2$ is among $q^{-1}, q^{-2}, \dots, q^{-n/2}$. Next assume that \mathcal{V} has X -type DDa. By Lemma 12.1 $k_0^2 = q^{-n-1}$. By Lemma 12.2 $k_0^2 k_3^2$ is not among $q^{-2}, q^{-4}, \dots, q^{-2n}$. By these comments $k_3^2 q^{-n-1}$ is not among $q^{-2}, q^{-4}, \dots, q^{-2n}$. So neither of $\pm k_3$ is among $q^{(n-1)/2}, q^{(n-3)/2}, \dots, q, 1, q^{-1}, \dots, q^{(1-n)/2}$. So none of $\pm k_3^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$. By Lemma 12.3 none of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, \dots, q^{-n}$. For the remaining types, we can show the result in a similar way as the case DDa. \square

13 A basis $\{u_r\}_{r=0}^n$ for \mathcal{V} on which X is upper tridiagonal and Y is lower tridiagonal

Throughout this section Notation 10.1 is in effect. In this section we introduce a certain basis $\{u_r\}_{r=0}^n$ for \mathcal{V} . A square matrix is said to be *upper tridiagonal* whenever each nonzero entry lies on the diagonal, the superdiagonal, or immediately above the superdiagonal. A square matrix is said to be *lower tridiagonal* whenever its transpose is upper tridiagonal. In later sections we show that with respect to the basis $\{u_r\}_{r=0}^n$ the matrix representing X is upper tridiagonal and the matrix representing Y is lower tridiagonal. To define the basis $\{u_r\}_{r=0}^n$, we consider the following scalars.

Definition 13.1 For $r = 0, 1, 2, \dots$ we define $\beta_r \in \mathbb{F}$ as follows:

X -type of \mathcal{V}	Definition of β_r
DS, DDa	$\beta_r = \begin{cases} k_0 k_1 q^r & \text{if } r \text{ is even} \\ k_0 k_1 q^{r+1} & \text{if } r \text{ is odd} \end{cases}$
DDb	$\beta_r = \begin{cases} (k_2 k_3 q^{r+1})^{-1} & \text{if } r \text{ is even} \\ (k_2 k_3 q^r)^{-1} & \text{if } r \text{ is odd} \end{cases}$
SSa	$\beta_r = \begin{cases} (k_0 k_1 q^r)^{-1} & \text{if } r \text{ is even} \\ (k_0 k_1 q^{r+1})^{-1} & \text{if } r \text{ is odd} \end{cases}$
SSb	$\beta_r = \begin{cases} k_2 k_3 q^{r+1} & \text{if } r \text{ is even} \\ k_2 k_3 q^r & \text{if } r \text{ is odd} \end{cases}$

Definition 13.2 Pick $0 \neq u_0 \in \mathcal{V}_X(\mu_0)$. We define vectors u_1, u_2, \dots in \mathcal{V} inductively as follows:

$$u_r = \begin{cases} u_{r-1} - \beta_{r-1} Y u_{r-1} & \text{if } \mu_{r-1} \text{ and } \mu_r \text{ are 1-adjacent,} \\ u_{r-1} - \beta_{r-1} Y^{-1} u_{r-1} & \text{if } \mu_{r-1} \text{ and } \mu_r \text{ are } q\text{-adjacent.} \end{cases}$$

Lemma 13.3 *With reference to Definition 13.2, the vectors $\{u_r\}_{r=0}^n$ form a basis for \mathcal{V} .*

Proof. Let $\{v_r\}_{r=0}^n$ denote the X -standard basis for \mathcal{V} corresponding to the ordering $\{\mu_r\}_{r=0}^n$. For notational convenience set $\mathcal{V}_r = \mathcal{V}_X(\mu_r)$ for $0 \leq r \leq n$ and $\mathcal{V}_{-1} = 0$. First assume that \mathcal{V} has X -type DS. By Lemma 8.6 $t_0 v_0 \in \mathcal{V}_0$ and $t_0^{-1} v_0 \in \mathcal{V}_0$. By Lemma 9.1

$$t_0 v_r \in \begin{cases} \mathcal{V}_{r-1} + \mathcal{V}_r & \text{if } r \text{ is even,} \\ (\mathcal{V}_r + \mathcal{V}_{r+1}) \setminus \mathcal{V}_r & \text{if } r \text{ is odd} \end{cases} \quad (1 \leq r \leq n).$$

By this and (4)

$$t_0^{-1} v_r \in \begin{cases} \mathcal{V}_{r-1} + \mathcal{V}_r & \text{if } r \text{ is even,} \\ (\mathcal{V}_r + \mathcal{V}_{r+1}) \setminus \mathcal{V}_r & \text{if } r \text{ is odd} \end{cases} \quad (1 \leq r \leq n).$$

Similarly, $t_1 v_n \in \mathcal{V}_n$, $t_1^{-1} v_n \in \mathcal{V}_n$, and

$$\begin{aligned} t_1 v_r &\in \begin{cases} (\mathcal{V}_r + \mathcal{V}_{r+1}) \setminus \mathcal{V}_r & \text{if } r \text{ is even,} \\ \mathcal{V}_{r-1} + \mathcal{V}_r & \text{if } r \text{ is odd} \end{cases} & (0 \leq r \leq n-1), \\ t_1^{-1} v_r &\in \begin{cases} (\mathcal{V}_r + \mathcal{V}_{r+1}) \setminus \mathcal{V}_r & \text{if } r \text{ is even,} \\ \mathcal{V}_{r-1} + \mathcal{V}_r & \text{if } r \text{ is odd} \end{cases} & (0 \leq r \leq n-1). \end{aligned}$$

Using these comments one routinely finds that for $0 \leq r \leq n-2$

$$\begin{aligned} Y v_r &\in (\mathcal{V}_{r-1} + \mathcal{V}_r + \mathcal{V}_{r+1} + \mathcal{V}_{r+2}) \setminus (\mathcal{V}_{r-1} + \mathcal{V}_r + \mathcal{V}_{r+1}) & \text{if } r \text{ is even,} \\ Y^{-1} v_r &\in (\mathcal{V}_{r-1} + \mathcal{V}_r + \mathcal{V}_{r+1} + \mathcal{V}_{r+2}) \setminus (\mathcal{V}_{r-1} + \mathcal{V}_r + \mathcal{V}_{r+1}) & \text{if } r \text{ is odd.} \end{aligned} \quad (70)$$

We now show that $\{u_r\}_{r=0}^n$ are linearly independent. To this end we show

$$u_r \in (\mathcal{V}_0 + \mathcal{V}_1 + \cdots + \mathcal{V}_r) \setminus (\mathcal{V}_0 + \mathcal{V}_1 + \cdots + \mathcal{V}_{r-1}) \quad (71)$$

using induction on $r = 0, 1, \dots, n$. For $r = 0$ the line (71) obviously holds. For $r = 1$ we argue as follows. By Definition 13.2 and the shape of the X -diagram, $u_1 = u_0 - \beta_0 Y^{-1} u_0$. We have $u_0 \in \mathcal{V}_0$ and $t_0^{-1} u_0 \in \mathcal{V}_0$. We have $t_1^{-1} v_0 \in (\mathcal{V}_0 + \mathcal{V}_1) \setminus \mathcal{V}_0$. So $Y^{-1} v_0 \in (\mathcal{V}_0 + \mathcal{V}_1) \setminus \mathcal{V}_0$. By these comments $u_1 \in (\mathcal{V}_0 + \mathcal{V}_1) \setminus \mathcal{V}_0$, and so (71) holds for $r = 1$. Now assume $2 \leq r \leq n$. First assume that r is even. By Definition 13.2 and the shape of the X -diagram,

$$u_r = u_{r-1} - \beta_{r-1} Y u_{r-1}, \quad u_{r-1} = u_{r-2} - \beta_{r-2} Y^{-1} u_{r-2}.$$

Combining these two equations,

$$u_r = u_{r-1} + \beta_{r-1} \beta_{r-2} u_{r-2} - \beta_{r-1} Y u_{r-2}.$$

By induction

$$\begin{aligned} u_{r-1} &\in \mathcal{V}_0 + \cdots + \mathcal{V}_{r-1}, \\ u_{r-2} &\in (\mathcal{V}_0 + \cdots + \mathcal{V}_{r-2}) \setminus (\mathcal{V}_0 + \cdots + \mathcal{V}_{r-3}). \end{aligned}$$

By (70)

$$Yv_{r-2} \in (\mathcal{V}_{r-3} + \mathcal{V}_{r-2} + \mathcal{V}_{r-1} + \mathcal{V}_r) \setminus (\mathcal{V}_{r-3} + \mathcal{V}_{r-2} + \mathcal{V}_{r-1}).$$

By these comments (71) holds for even r . Next assume that r is odd. We can show (71) in a similar way. Thus the vectors $\{u_r\}_{r=0}^n$ are linearly independent, and so form a basis for \mathcal{V} . We have shown the result when \mathcal{V} has X -type DS. The proof is similar for the other types. \square

Lemma 13.4 *With reference to Definition 13.2, $u_r = 0$ for $r > n$.*

Proof. Let $\{v_r\}_{r=0}^n$ denote the X -standard basis for \mathcal{V} such that $v_0 = u_0$. By Lemmas 8.6, 9.1, 9.2 we have the actions of $\{t_i\}_{i \in \mathbb{I}}$ on $\{v_r\}_{r=0}^n$. Using these actions and Definition 13.2, we represent u_r inductively for $r = 0, 1, \dots, n+1$ as a linear combination of $\{v_r\}_{r=0}^n$. Here we omit the precise computations. Now using Lemma 12.1 we find that $u_{n+1} = 0$. The result follows from this and Definition 13.2. \square

14 The action of $Y^{\pm 1}$ on the basis $\{u_r\}_{r=0}^n$

Throughout this section Notation 10.1 is in effect. Let the scalars β_0, β_1, \dots be from Definition 13.1 and the vectors u_0, u_1, \dots be from Definition 13.2. By Lemma 13.3 $\{u_r\}_{r=0}^n$ form a basis for \mathcal{V} . In this section we obtain the action of $Y^{\pm 1}$ on the basis $\{u_r\}_{r=0}^n$, and show that with respect to the basis $\{u_r\}_{r=0}^n$ the matrix representing $Y^{\pm 1}$ is lower tridiagonal. As a corollary, we obtain the condition for that Y is diagonalizable on \mathcal{V} . Note that $u_r = 0$ for $r > n$ by Lemma 13.4.

Lemma 14.1 *Assume that \mathcal{V} has X -type among DS, DDa. Then for $0 \leq r \leq n$*

$$Yu_r = \begin{cases} \beta_r u_r + \beta_{r+2}^{-1}(u_{r+1} - u_{r+2}) & \text{if } r \text{ is even,} \\ \beta_r^{-1}(u_r - u_{r+1}) & \text{if } r \text{ is odd,} \end{cases} \quad (72)$$

$$Y^{-1}u_r = \begin{cases} \beta_r^{-1}(u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ \beta_r u_r + \beta_r^{-1}(u_{r+1} - u_{r+2}) & \text{if } r \text{ is odd.} \end{cases} \quad (73)$$

Proof. Note by Definition 13.1 that

$$\beta_r = \begin{cases} k_0 k_1 q^r & \text{if } r \text{ is even,} \\ k_0 k_1 q^{r+1} & \text{if } r \text{ is odd} \end{cases} \quad (r = 0, 1, 2, \dots).$$

Pick any integer r such that $0 \leq r \leq n$. By Definition 13.2

$$u_{r+1} = u_r - k_0 k_1 q^{r+1} Y u_r \quad \text{if } r \text{ is odd.} \quad (74)$$

So (72) holds for odd r . By Definition 13.2

$$u_{r+1} = u_r - k_0 k_1 q^r Y^{-1} u_r \quad \text{if } r \text{ is even.} \quad (75)$$

So (73) holds for even r . Applying Y^{-1} to (74)

$$Y^{-1}u_{r+1} = Y^{-1}u_r - k_0k_1q^{r+1}u_r \quad \text{if } r \text{ is odd.}$$

By (73)

$$Y^{-1}u_{r+1} = (k_0k_1q^{r+1})^{-1}(u_{r+1} - u_{r+2}) \quad \text{if } r \text{ is odd.}$$

Combining the above two equations, we obtain (73) for odd r . Applying Y to (75)

$$Yu_{r+1} = Yu_r - k_0k_1q^ru_r \quad \text{if } r \text{ is even.}$$

By (72)

$$Yu_{r+1} = (k_0k_1q^{r+2})^{-1}(u_{r+1} - u_{r+2}) \quad \text{if } r \text{ is even.}$$

Combining the above two equations we obtain (72) for even r . \square

The following three lemmas can be shown in a similar way.

Lemma 14.2 *Assume that \mathcal{V} has X -type DDb. Then for $0 \leq r \leq n$*

$$Yu_r = \begin{cases} \beta_ru_r + \beta_r^{-1}(u_{r+1} - u_{r+2}) & \text{if } r \text{ is even,} \\ \beta_r^{-1}(u_r - u_{r+1}) & \text{if } r \text{ is odd,} \end{cases}$$

$$Y^{-1}u_r = \begin{cases} \beta_r^{-1}(u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ \beta_ru_r + \beta_{r+2}^{-1}(u_{r+1} - u_{r+2}) & \text{if } r \text{ is odd.} \end{cases}$$

Lemma 14.3 *Assume that \mathcal{V} has X -type SSa. Then for $0 \leq r \leq n$*

$$Yu_r = \begin{cases} \beta_r^{-1}(u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ \beta_ru_r + \beta_r^{-1}(u_{r+1} - u_{r+2}) & \text{if } r \text{ is odd,} \end{cases}$$

$$Y^{-1}u_r = \begin{cases} \beta_ru_r + \beta_{r+2}^{-1}(u_{r+1} - u_{r+2}) & \text{if } r \text{ is even,} \\ \beta_r^{-1}(u_r - u_{r+1}) & \text{if } r \text{ is odd.} \end{cases}$$

Lemma 14.4 *Assume that \mathcal{V} has X -type SSb. Then for $0 \leq r \leq n$*

$$Yu_r = \begin{cases} \beta_r^{-1}(u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ \beta_ru_r + \beta_{r+2}^{-1}(u_{r+1} - u_{r+2}) & \text{if } r \text{ is odd,} \end{cases}$$

$$Y^{-1}u_r = \begin{cases} \beta_ru_r + \beta_r^{-1}(u_{r+1} - u_{r+2}) & \text{if } r \text{ is even,} \\ \beta_r^{-1}(u_r - u_{r+1}) & \text{if } r \text{ is odd.} \end{cases}$$

By Lemmas 14.1–14.4 we obtain the following result.

Lemma 14.5 *With respect to the basis $\{u_r\}_{r=0}^n$ the matrix representing Y is lower tridiagonal with the following diagonal entries:*

X -type of \mathcal{V}	Diagonal entries
DS	$\beta_0, \beta_1^{-1}, \beta_2, \beta_3^{-1}, \dots, \beta_{n-1}^{-1}, \beta_n$
DDa, DDb	$\beta_0, \beta_1^{-1}, \beta_2, \beta_3^{-1}, \dots, \beta_{n-1}, \beta_n^{-1}$
SSa, SSb	$\beta_0^{-1}, \beta_1, \beta_2^{-1}, \beta_3, \dots, \beta_{n-1}^{-1}, \beta_n$

(76)

Proposition 14.6 *There exists a basis for \mathcal{V} with respect to which the matrix representing Y is lower tridiagonal with the following diagonal entries:*

X -type of \mathcal{V}	Diagonal entries
DS	$k_0 k_1, \frac{1}{k_0 k_1 q^2}, k_0 k_1 q^2, \frac{1}{k_0 k_1 q^4}, \dots, \frac{1}{k_0 k_1 q^n}, k_0 k_1 q^n$
DDa, SSa	$k_0 k_1, \frac{1}{k_0 k_1 q^2}, k_0 k_1 q^2, \frac{1}{k_0 k_1 q^4}, \dots, k_0 k_1 q^{n-1}, \frac{1}{k_0 k_1 q^{n+1}}$
DDb, SSb	$\frac{1}{k_2 k_3 q}, k_2 k_3 q, \frac{1}{k_2 k_3 q^3}, k_2 k_3 q^3, \dots, \frac{1}{k_2 k_3 q^n}, k_2 k_3 q^n$

Proof. Follows from Definition 13.1 and Lemma 14.5. □

Lemma 14.7 *Y is multiplicity-free on any $YD \hat{H}_q$ -module.*

Proof. By Proposition 7.8 X is multiplicity-free on any $XD \hat{H}_q$ -module. Consider the automorphism σ from Lemma 3.3. By Lemma 3.5 σ sends $Y \mapsto X$. The result follows. □

Corollary 14.8 *The following (i)–(iii) are equivalent:*

- (i) Y is diagonalizable on \mathcal{V} ;
- (ii) Y is multiplicity-free on \mathcal{V} ;
- (iii) the following inequalities hold:

X -type of \mathcal{V}	Inequalities
DS, DDa, SSa	Neither of $\pm k_0 k_1$ is among $q^{-1}, q^{-2}, q^{-3}, \dots, q^{-n}$
DDb, SSb	Neither of $\pm k_2 k_3$ is among $q^{-1}, q^{-2}, q^{-3}, \dots, q^{-n}$

(77)

Proof. (i) \Rightarrow (ii): Follows from Lemma 14.7.

(ii) \Rightarrow (i): By the definition.

(ii) \Leftrightarrow (iii): One verifies that the eigenvalues of Y given in Proposition 14.6 are mutually distinct if and only if (77) holds. □

Note 14.9 The action of \mathbb{A} on $\{u_r\}_{r=0}^n$ is immediately obtained from the action of $Y^{\pm 1}$ given in Lemmas 14.1–14.4. Observe that with respect to the basis $\{u_r\}_{r=0}^n$ the matrix representing \mathbb{A} is lower tridiagonal. Moreover it is not lower bidiagonal when $n \geq 2$.

15 The action of $\{t_i\}_{i \in \mathbb{I}}$ on the basis $\{u_r\}_{r=0}^n$

Throughout this section Notation 10.1 is in effect. Let the vectors u_0, u_1, \dots be from Definition 13.2. By Lemma 13.3 $\{u_r\}_{r=0}^n$ form a basis for \mathcal{V} . Note that $u_r = 0$ for $r > n$ by Lemma 13.4. For notational convenience, define $u_r = 0$ for $r < 0$. In this section we obtain the action of $\{t_i\}_{i \in \mathbb{I}}$ on $\{u_r\}_{r=0}^n$.

Lemma 15.1 *Assume that \mathcal{V} has X -type DS. Then for $0 \leq r \leq n$*

$$t_0 u_r = \begin{cases} k_0 q^r u_{r-1} + \left(\frac{1}{k_0 q^r} - k_0 - k_0^{-1}\right) (u_{r-1} - u_r) & \text{if } r \text{ is even,} \\ \frac{1}{k_0 q^{r+1}} (u_r - u_{r+1}) & \text{if } r \text{ is odd,} \end{cases} \quad (78)$$

$$t_1 u_r = \begin{cases} -k_1(1 - q^r)(1 - k_0^2 q^r) u_{r-1} + k_1 u_r + k_1^{-1} u_{r+1} & \text{if } r \text{ is even,} \\ k_1^{-1} u_r & \text{if } r \text{ is odd.} \end{cases} \quad (79)$$

Proof. We show (78) and (79) using induction on $r = 0, 1, \dots, n$. By Definition 8.5 $t_0 u_0 = k_0 u_0$. So (78) holds for $r = 0$. By Definition 13.2

$$u_1 = u_0 - k_0 k_1 t_1^{-1} t_0^{-1} u_0.$$

In this line, use $t_0^{-1} u_0 = k_0^{-1} u_0$, $t_1^{-1} = T_1 - t_1$ and (4) to find

$$t_1 u_0 = k_1 u_0 + k_1^{-1} u_1.$$

So (79) holds for $r = 0$. Now pick any integer r such that $1 \leq r \leq n$. First assume that r is odd. By Definition 13.2

$$u_r = u_{r-1} - k_0 k_1 q^{r-1} t_1^{-1} t_0^{-1} u_{r-1}.$$

In this line, apply t_1 to each side, use $t_0^{-1} = T_0 - t_0$ and (4), and evaluate $t_1 u_{r-1}$ and $t_0 u_{r-1}$ by induction. This yields (79) for odd r . By $t_0 = Y t_1^{-1}$, $t_1^{-1} = T_1 - t_1$ and (4)

$$t_0 u_r = (k_1 + k_1^{-1}) Y u_r - Y t_1 u_r.$$

In this line, use $t_1 u_r = k_1^{-1} u_r$, and evaluate $Y u_r$ by (72). This yields (78) for odd r . Next assume that r is even. By Definition 13.2

$$u_r = u_{r-1} - k_0 k_1 q^r t_0 t_1 u_{r-1}.$$

In this line, evaluate $t_1 u_{r-1}$ by induction to find

$$u_r = u_{r-1} - k_0 q^r t_0 u_{r-1}.$$

In this line, apply t_0 to each side, and use $t_0^2 = t_0 T_0 - 1$ and (4) to find

$$t_0 u_r = t_0 u_{r-1} - k_0 q^r (k_0 + k_0^{-1}) t_0 u_{r-1} + k_0 q^r u_{r-1}.$$

In this line, evaluate $t_0 u_{r-1}$ by induction. This yields (78) for even r . By $t_1 = T_1 - t_1^{-1}$, $t_1^{-1} = Y^{-1} t_0$ and (4)

$$t_1 u_r = (k_1 + k_1^{-1}) u_r - Y^{-1} t_0 u_r.$$

In this line, use (78) for even r and (73). This yields (79) for even r . \square

Lemma 15.2 Assume that \mathcal{V} has X -type DS. Then for $0 \leq r \leq n$

$$t_2 u_r = \begin{cases} k_2 q^{n-r} (u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ \frac{(1-q^{n-r+1})(1-k_2^2 q^{n-r+1})}{k_2 q^{n-r+1}} u_{r-1} + (k_2 + k_2^{-1} - k_2 q^{n-r+1}) u_r & \text{if } r \text{ is odd,} \end{cases} \quad (80)$$

$$t_3 u_r = \begin{cases} k_3 u_r & \text{if } r \text{ is even,} \\ -\frac{(1-q^{n-r+1})(1-k_2^2 q^{n-r+1})}{k_2^2 k_3 q^{2(n-r+1)}} u_{r-1} + k_3^{-1} u_r + k_3 u_{r+1} & \text{if } r \text{ is odd.} \end{cases} \quad (81)$$

Proof. Note that $k_0 k_1 k_2 k_3 = q^{-n-1}$ by Lemma 12.1. Also note by (3) that $Y = q^{-1} t_3^{-1} t_2^{-1}$ and $Y^{-1} = q t_2 t_3$. We show (80) and (81) using induction on $r = 0, 1, \dots, n$. By Definition 8.5 $t_3 u_0 = k_3 u_0$. So (81) holds for $r = 0$. By Definition 13.2

$$u_1 = u_0 - k_0 k_1 Y^{-1} u_0.$$

In this line, use $Y^{-1} = q t_2 t_3$, $t_3 u_0 = k_3 u_0$ and $k_0 k_1 k_3 q = k_2^{-1} q^{-n}$ to find

$$t_2 u_0 = k_2 q^n (u_0 - u_1).$$

So (80) holds for $r = 0$. Now pick any integer r such that $1 \leq r \leq n$. First assume that r is odd. By Definition 13.2

$$u_r = u_{r-1} - k_0 k_1 q^{r-1} Y^{-1} u_{r-1}.$$

In this line, use $Y^{-1} = q t_2 t_3$, and evaluate $t_3 u_{r-1}$ by induction to find

$$u_r = u_{r-1} - k_0 k_1 k_3 q^r t_2 u_{r-1}.$$

In this line, apply t_2^{-1} to each side to find

$$t_2^{-1} u_r = t_2^{-1} u_{r-1} - k_0 k_1 k_3 q^r u_{r-1}.$$

In this line, use $t_2^{-1} = T_2 - t_2$ and (4) to find

$$t_2 u_r = t_2 u_{r-1} + (k_2 + k_2^{-1})(u_r - u_{r+1}) + k_0 k_1 k_3 q^r u_{r-1}.$$

In this line, evaluate $t_2 u_{r-1}$ by induction, and use $k_0 k_1 k_3 = q^{-n-1} k_2^{-1}$. This yields (80) for odd r . By $t_3 = T_3 - t_3^{-1}$, $t_3^{-1} = q Y t_2$ and (4)

$$t_3 u_r = (k_3 + k_3^{-1}) u_r - q Y t_2 u_r.$$

In this line, use (80) for odd r , (72), and $k_3 = q^{-n-1} (k_0 k_1 k_2)^{-1}$. This yields (81) for odd r . Next assume that r is even. By Definition 13.2

$$u_r = u_{r-1} - k_0 k_1 q^r Y u_{r-1}.$$

In this line, apply t_3 to each side, and use $Y = q^{-1} t_3^{-1} t_2^{-1}$, $t_2^{-1} = T_2 - t_2$ and (4) to find

$$t_3 u_r = t_3 u_{r-1} - k_0 k_1 q^{r-1} (k_2 + k_2^{-1}) u_{r-1} + k_0 k_1 q^{r-1} t_2 u_{r-1}.$$

In this line, evaluate $t_3 u_{r-1}$ and $t_2 u_{r-1}$ by induction. This yields (81) for even r . By $t_2 = q^{-1} Y^{-1} t_3^{-1}$, $t_3^{-1} = T_3 - t_3$ and (4)

$$t_2 u_r = q^{-1} Y^{-1} (k_3 + k_3^{-1}) u_r - q^{-1} Y^{-1} t_3 u_r.$$

In this line, use (81) for even r , (73), and $k_0 k_1 k_3 = q^{-n-1} k_2^{-1}$. This yields (80) holds for even r . \square

We have obtained the action of $\{t_i\}_{i \in \mathbb{I}}$ when \mathcal{V} has X -type DS. In a similar way, we obtain the following four lemmas.

Lemma 15.3 *Assume that \mathcal{V} has X -type DDa. Then for $0 \leq r \leq n$*

$$\begin{aligned} t_0 u_r &= \begin{cases} -\frac{(1-q^r)(1-q^{n-r+1})}{k_0 q^{n+1}} u_{r-1} + \left(k_0 + k_0^{-1} - \frac{1}{k_0 q^r}\right) u_r & \text{if } r \text{ is even,} \\ \frac{1}{k_0 q^{r+1}} (u_r - u_{r+1}) & \text{if } r \text{ is odd,} \end{cases} \\ t_1 u_r &= \begin{cases} \frac{(1-q^r)(1-q^{n-r+1})}{k_1^{-1} q^{n-r+1}} u_{r-1} + k_1 u_r + k_1^{-1} u_{r+1} & \text{if } r \text{ is even,} \\ k_1^{-1} u_r & \text{if } r \text{ is odd,} \end{cases} \\ t_2 u_r &= \begin{cases} \frac{1}{k_0 k_1 k_3 q^{r+1}} (u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ \frac{(1-k_0 k_1 k_2 k_3 q^r)(1-k_0 k_1 k_2^{-1} k_3 q^r)}{k_0 k_1 k_3 q^r} u_{r-1} + \left(k_2 + k_2^{-1} - \frac{1}{k_0 k_1 k_3 q^r}\right) u_r & \text{if } r \text{ is odd,} \end{cases} \\ t_3 u_r &= \begin{cases} k_3 u_r & \text{if } r \text{ is even,} \\ -k_3^{-1} (1 - k_0 k_1 k_2 k_3 q^r) (1 - k_0 k_1 k_2^{-1} k_3 q^r) u_{r-1} + k_3^{-1} u_r + k_3 u_{r+1} & \text{if } r \text{ is odd.} \end{cases} \end{aligned}$$

Lemma 15.4 *Assume that \mathcal{V} has X -type DDb. Then for $0 \leq r \leq n$*

$$\begin{aligned} t_0 u_r &= \begin{cases} k_0 u_r & \text{if } r \text{ is even,} \\ \frac{(1-k_0 k_1 k_2 k_3 q^r)(1-k_0 k_1^{-1} k_2 k_3 q^r)}{k_0 k_2^2 q^{2r-n-1}} u_{r-1} + k_0^{-1} (u_r - u_{r+1}) & \text{if } r \text{ is odd,} \end{cases} \\ t_1 u_r &= \begin{cases} (k_1 + k_1^{-1} - k_0 k_2 k_3 q^{r+1}) u_r + k_0 k_2 k_3 q^{r+1} u_{r+1} & \text{if } r \text{ is even,} \\ \left(k_1 + k_1^{-1} - k_0 k_2 k_3 q^r - \frac{1}{k_0 k_2 k_3 q^r}\right) u_{r-1} + k_0 k_2 k_3 q^r u_r & \text{if } r \text{ is odd,} \end{cases} \\ t_2 u_r &= \begin{cases} -\frac{(1-q^r)(1-q^{n-r+1})}{k_2 q^r} u_{r-1} + k_2 (u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ k_2^{-1} u_r & \text{if } r \text{ is odd,} \end{cases} \\ t_3 u_r &= \begin{cases} \left(k_3 + k_3^{-1} - k_3 q^r - \frac{1}{k_3 q^r}\right) u_{r-1} + k_3 q^r u_r & \text{if } r \text{ is even,} \\ (k_3 + k_3^{-1} - k_3 q^{r+1}) u_r + k_3 q^{r+1} u_{r+1} & \text{if } r \text{ is odd.} \end{cases} \end{aligned}$$

Lemma 15.5 Assume that \mathcal{V} has X -type SSa. Then for $0 \leq r \leq n$

$$\begin{aligned}
t_0 u_r &= \begin{cases} -\frac{(1-q^r)(1-q^{n-r+1})}{k_0 q^r} u_{r-1} + k_0(u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ k_0^{-1} u_r & \text{if } r \text{ is odd,} \end{cases} \\
t_1 u_r &= \begin{cases} (k_1 + k_1^{-1} - k_1 q^r - \frac{1}{k_1 q^r}) u_{r-1} + k_1 q^r u_r & \text{if } r \text{ is even,} \\ (k_1 + k_1^{-1} - k_1 q^{r+1}) u_r + k_1 q^{r+1} u_{r+1} & \text{if } r \text{ is odd,} \end{cases} \\
t_2 u_r &= \begin{cases} k_2 u_r & \text{if } r \text{ is even,} \\ \frac{(1-k_0 k_1 k_2 k_3 q^r)(1-k_0 k_1 k_2 k_3^{-1} q^r)}{k_0^2 k_2 q^{2r-n-1}} u_{r-1} + k_2^{-1}(u_r - u_{r+1}) & \text{if } r \text{ is odd,} \end{cases} \\
t_3 u_r &= \begin{cases} (k_3 + k_3^{-1} - k_0 k_1 k_2 q^{r+1}) u_r + k_0 k_1 k_2 q^{r+1} u_{r+1} & \text{if } r \text{ is even,} \\ (k_3 + k_3^{-1} - k_0 k_1 k_2 q^r - \frac{1}{k_0 k_1 k_2 q^r}) u_{r-1} + k_0 k_1 k_2 q^r u_r & \text{if } r \text{ is odd.} \end{cases}
\end{aligned}$$

Lemma 15.6 Assume that \mathcal{V} has X -type SSb. Then for $0 \leq r \leq n$

$$\begin{aligned}
t_0 u_r &= \begin{cases} \frac{1}{k_1 k_2 k_3 q^{r+1}} (u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ (k_1 k_2 k_3 q^r + \frac{1}{k_1 k_2 k_3 q^r} - k_0 - k_0^{-1}) u_{r-1} + (k_0 + k_0^{-1} - \frac{1}{k_1 k_2 k_3 q^r}) u_r & \text{if } r \text{ is odd,} \end{cases} \\
t_1 u_r &= \begin{cases} k_1 u_r & \text{if } r \text{ is even,} \\ -k_1^{-1}(1 - k_0 k_1 k_2 k_3 q^r)(1 - k_0^{-1} k_1 k_2 k_3 q^r) u_{r-1} + k_1^{-1} u_r + k_1 u_{r+1} & \text{if } r \text{ is odd,} \end{cases} \\
t_2 u_r &= \begin{cases} -\frac{(1-q^r)(1-q^{n-r+1})}{k_2 q^{n+1}} u_{r-1} + (k_2 + k_2^{-1} - \frac{1}{k_2 q^r}) u_r & \text{if } r \text{ is even,} \\ \frac{1}{k_2 q^{r+1}} (u_r - u_{r+1}) & \text{if } r \text{ is odd,} \end{cases} \\
t_3 u_r &= \begin{cases} \frac{(1-q^r)(1-q^{n-r+1})}{k_3^{-1} q^{n-r+1}} u_{r-1} + k_3 u_r + k_3^{-1} u_{r+1} & \text{if } r \text{ is even,} \\ k_3^{-1} u_r & \text{if } r \text{ is odd.} \end{cases}
\end{aligned}$$

16 The action of $X^{\pm 1}$ on the basis $\{u_r\}_{r=0}^n$

Throughout this section Notation 10.1 is in effect. Consider the basis $\{u_r\}_{r=0}^n$ from Definition 13.2. In this section we obtain the action of $X^{\pm 1}$ on $\{u_r\}_{r=0}^n$, and show that with respect to the basis $\{u_r\}_{r=0}^n$ the matrix representing $X^{\pm 1}$ is upper tridiagonal.

To simplify the action of $X^{\pm 1}$, we normalize the vectors $\{u_r\}_{r=0}^n$ using the following scalars.

Definition 16.1 Set $e_0 = 1$. For $1 \leq r \leq n$ define $e_r \in \mathbb{F}$ as follows:

X -type of \mathcal{V}	e_r
DS, DDa	$\begin{cases} \frac{1}{(1-q^r)(1-k_0^2 q^r)} & \text{if } r \text{ is even} \\ \frac{1}{(1-k_0 k_1 k_2 k_3 q^r)(1-k_0 k_1 k_2^{-1} k_3 q^r)} & \text{if } r \text{ is odd} \end{cases}$
DDb	$\begin{cases} -\frac{1}{k_0^2(1-q^r)(1-k_3^2 q^r)} & \text{if } r \text{ is even} \\ -\frac{k_2^2}{(1-k_0 k_1 k_2 k_3 q^r)(1-k_0 k_1^{-1} k_2 k_3 q^r)} & \text{if } r \text{ is odd} \end{cases}$
SSa	$\begin{cases} -\frac{1}{k_2^2(1-q^r)(1-k_1^2 q^r)} & \text{if } r \text{ is even} \\ -\frac{k_0^2}{(1-k_0 k_1 k_2 k_3 q^r)(1-k_0 k_1 k_2 k_3^{-1} q^r)} & \text{if } r \text{ is odd} \end{cases}$
SSb	$\begin{cases} \frac{1}{(1-q^r)(1-k_2^2 q^r)} & \text{if } r \text{ is even} \\ \frac{1}{(1-k_0 k_1 k_2 k_3 q^r)(1-k_0^{-1} k_1 k_2 k_3 q^r)} & \text{if } r \text{ is odd} \end{cases}$

In the above, all the denominators are nonzero by Lemmas 12.1 and 12.4.

Definition 16.2 For $0 \leq r \leq n$ define

$$u'_r = e_0 e_1 \cdots e_r u_r.$$

Below we give formulas for the action of $X^{\pm 1}$ on the basis $\{u'_r\}_{r=0}^n$. For notational convenience, set $u'_r = 0$ for $r < 0$. These formulas can be routinely obtained using Lemmas 15.1–15.6.

Lemma 16.3 Assume that \mathcal{V} has X -type among DS, DDa. Then for $0 \leq r \leq n$

$$\begin{aligned} Xu'_r &= \begin{cases} k_0 k_3 q^r u'_r + (k_0 k_3 q^r)^{-1} (u'_{r-1} - u'_{r-2}) & \text{if } r \text{ is even,} \\ (k_0 k_3 q^{r+1})^{-1} (u'_r - u'_{r-1}) & \text{if } r \text{ is odd,} \end{cases} \\ X^{-1}u'_r &= \begin{cases} (k_0 k_3 q^r)^{-1} (u'_r - u'_{r-1}) & \text{if } r \text{ is even,} \\ k_0 k_3 q^{r+1} u'_r + (k_0 k_3 q^{r-1})^{-1} (u'_{r-1} - u'_{r-2}) & \text{if } r \text{ is odd.} \end{cases} \end{aligned}$$

Lemma 16.4 Assume that \mathcal{V} has X -type DDb. Then for $0 \leq r \leq n$

$$\begin{aligned} Xu'_r &= \begin{cases} k_0 k_3 q^r u'_r + (k_0 k_3 q^r)^{-1} u'_{r-1} & \text{if } r \text{ is even,} \\ (k_0 k_3 q^{r+1})^{-1} (u'_r - u'_{r-1}) - (k_0^3 k_3^3 q^{3r-1})^{-1} u'_{r-2} & \text{if } r \text{ is odd,} \end{cases} \\ X^{-1}u'_r &= \begin{cases} (k_0 k_3 q^r)^{-1} (u'_r - u'_{r-1}) - (k_0^3 k_3^3 q^{3r-2})^{-1} u'_{r-2} & \text{if } r \text{ is even,} \\ k_0 k_3 q^{r+1} u'_r + (k_0 k_3 q^{r-1})^{-1} u'_{r-1} & \text{if } r \text{ is odd.} \end{cases} \end{aligned}$$

Lemma 16.5 Assume that \mathcal{V} has X -type SSa. Then for $0 \leq r \leq n$

$$\begin{aligned} Xu'_r &= \begin{cases} (k_1 k_2 q^{r+1})^{-1} (u'_r - u'_{r-1}) - (k_1^3 k_2^3 q^{3r-1})^{-1} u'_{r-2} & \text{if } r \text{ is even,} \\ k_1 k_2 q^r u'_r + (k_1 k_2 q^r)^{-1} u'_{r-1} & \text{if } r \text{ is odd,} \end{cases} \\ X^{-1}u'_r &= \begin{cases} k_1 k_2 q^{r+1} u'_r + (k_1 k_2 q^{r-1})^{-1} u'_{r-1} & \text{if } r \text{ is even,} \\ (k_1 k_2 q^r)^{-1} (u'_r - u'_{r-1}) - (k_1^3 k_2^3 q^{3r-2})^{-1} u'_{r-2} & \text{if } r \text{ is odd.} \end{cases} \end{aligned}$$

Lemma 16.6 Assume that \mathcal{V} has X -type SSb. Then for $0 \leq r \leq n$

$$\begin{aligned} Xu'_r &= \begin{cases} (k_1 k_2 q^{r+1})^{-1} (u'_r - u'_{r-1}) & \text{if } r \text{ is even,} \\ k_1 k_2 q^r u'_r + (k_1 k_2 q^r)^{-1} (u'_{r-1} - u'_{r-2}) & \text{if } r \text{ is odd,} \end{cases} \\ X^{-1}u'_r &= \begin{cases} k_1 k_2 q^{r+1} u'_r + (k_1 k_2 q^{r-1})^{-1} (u'_{r-1} - u'_{r-2}) & \text{if } r \text{ is even,} \\ (k_1 k_2 q^r)^{-1} (u'_r - u'_{r-1}) & \text{if } r \text{ is odd.} \end{cases} \end{aligned}$$

Corollary 16.7 With respect to the basis $\{u_r\}_{r=0}^n$ the matrix representing $X^{\pm 1}$ is upper tridiagonal.

Proof. By lemmas 16.3–16.6 the matrix representing $X^{\pm 1}$ with respect to $\{u'_r\}_{r=0}^n$ is upper tridiagonal. The result follows since $\{u'_r\}_{r=0}^n$ is a normalization of $\{u_r\}_{r=0}^n$. \square

Note 16.8 The action of \mathbb{B} on $\{u'_r\}_{r=0}^n$ is immediately obtained from the action of $X^{\pm 1}$ given in Lemmas 16.3–16.6. Observe that with respect to the basis $\{u_r\}_{r=0}^n$ the matrix representing \mathbb{B} is upper tridiagonal. Moreover it is not upper bidiagonal when $n \geq 2$.

17 A basis for $\mathcal{V}(k_0^{\pm 1})$

Throughout this section Notation 10.1 is in effect. Assume $k_0 \neq k_0^{-1}$. Let the subspaces $\mathcal{V}(k_0^{\pm 1})$ be from (54), and the elements F^{\pm} be from (55). In this section, we construct a basis for $\mathcal{V}(k_0^{\pm 1})$ with respect to which the matrix representing \mathbb{A} is lower bidiagonal and the matrix representing \mathbb{B} is upper bidiagonal. Let the basis $\{u_r\}_{r=0}^n$ be from Definition 13.2. The following five Lemmas can be routinely obtained using the action of t_0 given in Lemmas 15.1–15.6.

Lemma 17.1 Assume that \mathcal{V} has X -type DS. Then for $0 \leq r \leq n$

$$F^+ u_r = \begin{cases} -\frac{1-k_0^2 q^r}{q^r(1-k_0^2)} ((1-q^r)u_{r-1} - u_r) & \text{if } r \text{ is even,} \\ -\frac{1}{q^{r+1}(1-k_0^2)} ((1-q^{r+1})u_r - u_{r+1}) & \text{if } r \text{ is odd,} \end{cases} \quad (82)$$

$$F^- u_r = \begin{cases} \frac{(1-q^r)}{q^r(1-k_0^2)} ((1-k_0^2 q^r)u_{r-1} - u_r) & \text{if } r \text{ is even,} \\ \frac{1}{q^{r+1}(1-k_0^2)} ((1-k_0^2 q^{r+1})u_r - u_{r+1}) & \text{if } r \text{ is odd.} \end{cases} \quad (83)$$

Lemma 17.2 Assume that \mathcal{V} has X -type DDa. Then for $0 \leq r \leq n$

$$F^+ u_r = \begin{cases} -\frac{1-q^{n-r+1}}{1-q^{n+1}} ((1-q^r)u_{r-1} - u_r) & \text{if } r \text{ is even,} \\ \frac{q^{n-r}}{1-q^{n+1}} ((1-q^{r+1})u_r - u_{r+1}) & \text{if } r \text{ is odd,} \end{cases} \quad (84)$$

$$F^- u_r = \begin{cases} \frac{1-q^r}{1-q^{n+1}} ((1-q^{n-r+1})u_{r-1} + q^{n-r+1}u_r) & \text{if } r \text{ is even,} \\ \frac{1}{1-q^{n+1}} ((1-q^{n-r})u_r + q^{n-r}u_{r+1}) & \text{if } r \text{ is odd.} \end{cases} \quad (85)$$

Lemma 17.3 Assume that \mathcal{V} has X -type DDb. Then for $0 \leq r \leq n$

$$F^+u_r = \begin{cases} u_r & \text{if } r \text{ is even,} \\ -\frac{(1-k_0k_1k_2k_3q^r)(1-k_0k_1^{-1}k_2k_3q^r)}{(1-k_0^2)k_2^2q^{2r-n-1}}u_{r-1} + \frac{1}{1-k_0^2}u_{r+1} & \text{if } r \text{ is odd,} \end{cases}$$

$$F^-u_r = \begin{cases} 0 & \text{if } r \text{ is even,} \\ \frac{(1-k_0k_1k_2k_3q^r)(1-k_0k_1^{-1}k_2k_3q^r)}{(1-k_0^2)k_2^2q^{2r-n-1}}u_{r-1} + u_r - \frac{1}{1-k_0^2}u_{r+1} & \text{if } r \text{ is odd.} \end{cases}$$

Lemma 17.4 Assume that \mathcal{V} has X -type SSa. Then for $0 \leq r \leq n$

$$F^+u_r = \begin{cases} \frac{(1-q^r)(1-q^{n-r+1})}{(1-k_0^2)q^r}u_{r-1} + u_r + \frac{k_0^2}{1-k_0^2}u_{r+1} & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd,} \end{cases}$$

$$F^-u_r = \begin{cases} -\frac{(1-q^r)(1-q^{n-r+1})}{(1-k_0^2)q^r}u_{r-1} - \frac{k_0^2}{1-k_0^2}u_{r+1} & \text{if } r \text{ is even,} \\ u_r & \text{if } r \text{ is odd.} \end{cases}$$

Lemma 17.5 Assume that \mathcal{V} has X -type SSb. Then for $0 \leq r \leq n$

$$F^+u_r = \begin{cases} -\frac{1}{(1-k_0^2)k_0^{-1}k_1k_2k_3q^{r+1}}((1-k_0^{-1}k_1k_2k_3q^{r+1})u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ -\frac{1-k_0k_1k_2k_3q^r}{(1-k_0^2)k_0^{-1}k_1k_2k_3q^r}((1-k_0^{-1}k_1k_2k_3q^r)u_{r-1} - u_r) & \text{if } r \text{ is odd,} \end{cases}$$

$$F^-u_r = \begin{cases} \frac{1}{(1-k_0^2)k_0^{-1}k_1k_2k_3q^{r+1}}((1-k_0k_1k_2k_3q^{r+1})u_r - u_{r+1}) & \text{if } r \text{ is even,} \\ \frac{1-k_0^{-1}k_1k_2k_3q^r}{(1-k_0^2)k_0^{-1}k_1k_2k_3q^r}((1-k_0k_1k_2k_3q^r)u_{r-1} - u_r) & \text{if } r \text{ is odd.} \end{cases}$$

We obtain a basis for $\mathcal{V}(k_0^{\pm 1})$ by applying F^{\pm} to an appropriate subset of $\{u_r\}_{r=0}^d$ as follows.

Lemma 17.6 The subspaces $\mathcal{V}(k_0^{\pm 1})$ have the following bases:

X -type of \mathcal{V}	Basis for $\mathcal{V}(k_0)$	Basis for $\mathcal{V}(k_0^{-1})$
DS	$\{F^+u_{2r}\}_{r=0}^{n/2}$	$\{F^-u_{2r}\}_{r=1}^{n/2}$
DDa	$\{F^+u_{2r}\}_{r=0}^{(n-1)/2} \cup \{F^+u_n\}$	$\{F^-u_{2r}\}_{r=1}^{(n-1)/2}$
DDb	$\{F^+u_{2r}\}_{r=0}^{(n-1)/2}$	$\{F^-u_{2r+1}\}_{r=0}^{(n-1)/2}$
SSa	$\{F^+u_{2r}\}_{r=0}^{(n-1)/2}$	$\{F^-u_{2r+1}\}_{r=0}^{(n-1)/2}$
SSb	$\{F^+u_{2r}\}_{r=0}^{(n-1)/2}$	$\{F^-u_{2r}\}_{r=0}^{(n-1)/2}$

Proof. First assume that \mathcal{V} has X -type DS. We first show that $\{F^+u_{2r}\}_{r=0}^{n/2}$ is a basis for $\mathcal{V}(k_0)$. By Corollary 10.11 the dimension of $\mathcal{V}(k_0)$ is $n/2 + 1$. So it suffices to show that $\{F^+u_{2r}\}_{r=0}^{n/2}$ are linearly independent. By Lemma 12.4 neither of $\pm k_0$ is among $q^{-1}, q^{-2}, \dots, q^{-n/2}$. So $k_0^2q^{2r} \neq 1$ for $1 \leq r \leq n/2$. By the assumption, $k_0^2 \neq 1$. By these comments $1 - k_0^2q^{2r}$ is nonzero for $0 \leq r \leq n/2$. By this and (82) F^+u_{2r} is nonzero for

$0 \leq r \leq n/2$. By this and (82) the vectors $\{F^+u_{2r}\}_{r=0}^{n/2}$ are linearly independent. Next we show that $\{F^-u_{2r}\}_{r=1}^{n/2}$ is a basis for $\mathcal{V}(k_0^{-1})$. By Corollary 10.11 the dimension of $\mathcal{V}(k_0^{-1})$ is $n/2$. So it suffices to show that $\{F^-u_{2r}\}_{r=1}^{n/2}$ are linearly independent. Observe by (83) that F^-u_{2r} is nonzero for $1 \leq r \leq n/2$. By this and (83) the vectors $\{F^-u_{2r}\}_{r=1}^{n/2}$ are linearly independent. We have shown the result when \mathcal{V} has X -type DS. The proof is similar for the other types. \square

18 The action of \mathbb{A} on $\mathcal{V}(k_0^{\pm 1})$

Throughout this section Notation 10.1 is in effect. Assume $k_0 \neq k_0^{-1}$. Let the subspaces $\mathcal{V}(k_0^{\pm 1})$ be from (54), and the elements F^\pm be from (55). Consider the basis (86) for $\mathcal{V}(k_0^{\pm 1})$. In this section, we obtain the action of \mathbb{A} on this basis. The following five Lemmas can be routinely obtained using Lemmas 14.1–14.4 and Lemmas 17.1–17.5.

Lemma 18.1 *Assume that \mathcal{V} has X -type DS. Then for $0 \leq r \leq n/2$*

$$\mathbb{A}F^+u_{2r} = \left(k_0k_1q^{2r} + \frac{1}{k_0k_1q^{2r}}\right)F^+u_{2r} - \frac{q^2(1 - k_0^2q^{2r})}{k_0k_1q^{2r+2}(1 - k_0^2q^{2r+2})}F^+u_{2r+2},$$

and for $1 \leq r \leq n/2$

$$\mathbb{A}F^-u_{2r} = \left(k_0k_1q^{2r} + \frac{1}{k_0k_1q^{2r}}\right)F^-u_{2r} - \frac{q^2(1 - q^{2r})}{k_0k_1q^{2r+2}(1 - q^{2r+2})}F^-u_{2r+2}.$$

Lemma 18.2 *Assume that \mathcal{V} has X -type DDa. Then for $0 \leq r \leq (n-3)/2$*

$$\begin{aligned}\mathbb{A}F^+u_{2r} &= \left(k_0k_1q^{2r} + \frac{1}{k_0k_1q^{2r}}\right)F^+u_{2r} - \frac{1 - q^{n-2r+1}}{k_0k_1q^{2r+2}(1 - q^{n-2r-1})}F^+u_{2r+2}, \\ \mathbb{A}F^+u_{n-1} &= \left(k_0k_1q^{n-1} + \frac{1}{k_0k_1q^{n-1}}\right)F^+u_{n-1} + \frac{1 - q^2}{k_0k_1q^{n+1}}F^+u_n, \\ \mathbb{A}F^+u_n &= \left(k_0k_1q^{n+1} + \frac{1}{k_0k_1q^{n+1}}\right)F^+u_n,\end{aligned}$$

and for $1 \leq r \leq (n-1)/2$

$$\mathbb{A}F^-u_{2r} = \left(k_0k_1q^{2r} + \frac{1}{k_0k_1q^{2r}}\right)F^-u_{2r} - \frac{1 - q^{2r}}{k_0k_1q^{2r}(1 - q^{2r+2})}F^-u_{2r+2}$$

Lemma 18.3 *Assume that \mathcal{V} has X -type DDb. Then for $0 \leq r \leq (n-1)/2$*

$$\begin{aligned}\mathbb{A}F^+u_{2r} &= \left(k_2k_3q^{2r+1} + \frac{1}{k_2k_3q^{2r+1}}\right)F^+u_{2r} - k_2k_3q^{2r+1}F^+u_{2r+2}, \\ \mathbb{A}F^-u_{2r+1} &= \left(k_2k_3q^{2r+1} + \frac{1}{k_2k_3q^{2r+1}}\right)F^-u_{2r+1} - k_2k_3q^{2r+3}F^-u_{2r+3}.\end{aligned}$$

Lemma 18.4 *Assume that \mathcal{V} has X -type SSa. Then for $0 \leq r \leq (n-1)/2$*

$$\begin{aligned}\mathbb{A}F^+u_{2r} &= \left(k_0k_1q^{2r} + \frac{1}{k_0k_1q^{2r}}\right)F^+u_{2r} - k_0k_1q^{2r+2}F^+u_{2r+2}, \\ \mathbb{A}F^-u_{2r+1} &= \left(k_0k_1q^{2r+2} + \frac{1}{k_0k_1q^{2r+2}}\right)F^-u_{2r+1} - k_0k_1q^{2r+2}F^-u_{2r+3}.\end{aligned}$$

Lemma 18.5 *Assume that \mathcal{V} has X -type SSb. Then for $0 \leq r \leq (n-1)/2$*

$$\begin{aligned}\mathbb{A}F^+u_{2r} &= \left(k_2k_3q^{2r+1} + \frac{1}{k_2k_3q^{2r+1}}\right)F^+u_{2r} - \frac{1}{k_2k_3q^{2r+1}}F^+u_{2r+2}, \\ \mathbb{A}F^-u_{2r} &= \left(k_2k_3q^{2r+1} + \frac{1}{k_2k_3q^{2r+1}}\right)F^-u_{2r} - \frac{1}{k_2k_3q^{2r+1}}F^-u_{2r+2}.\end{aligned}$$

By Lemmas 18.1–18.5 we obtain the following corollary.

Corollary 18.6 *With reference to Notation 10.9, the following hold.*

- (i) *Consider the basis for $\mathcal{V}(k_0)$ from (86). With respect to this basis the matrix representing \mathbb{A} is lower bidiagonal with the following (r, r) -entry for $0 \leq r \leq d$:*

X -type of \mathcal{V}	(r, r) -entry for \mathbb{A}	
DS, DDa, SSa	$k_0k_1q^{2r} + \frac{1}{k_0k_1q^{2r}}$	(87)
DDb, SSb	$k_2k_3q^{2r+1} + \frac{1}{k_2k_3q^{2r+1}}$	

- (ii) *Consider the basis for $\mathcal{V}(k_0^{-1})$ from (86). With respect to this basis the matrix representing \mathbb{A} is lower bidiagonal with the following (r, r) -entry for $0 \leq r \leq d'$:*

X -type of \mathcal{V}	(r, r) -entry for \mathbb{A}	
DS, DDa	$k_0k_1q^{2r+2} + \frac{1}{k_0k_1q^{2r+2}}$	(88)
DDb, SSb	$k_2k_3q^{2r+1} + \frac{1}{k_2k_3q^{2r+1}}$	
SSa	$k_0k_1q^{2r} + \frac{1}{k_0k_1q^{2r}}$	

19 The action of \mathbb{B} on $\mathcal{V}(k_0^{\pm 1})$

Throughout this section Notation 10.1 is in effect. Assume $k_0 \neq k_0^{-1}$. Let the subspaces $\mathcal{V}(k_0^{\pm 1})$ be from (54), and the elements F^{\pm} be from (55). Consider the basis (86) for $\mathcal{V}(k_0^{\pm 1})$. In this section we obtain the action of \mathbb{B} on this basis. We use the normalized basis $\{u'_r\}_{r=0}^n$ from Definition 16.2. The following five Lemmas can be routinely obtained using Lemmas 16.3–16.6 and 17.1–17.5. For notational convenience, set $u'_r = 0$ for $r < 0$.

Lemma 19.1 Assume that \mathcal{V} has X -type DS. Then for $0 \leq r \leq n/2$

$$\mathbb{B}F^+u'_{2r} = \left(k_0k_3q^{2r} + \frac{1}{k_0k_3q^{2r}}\right)F^+u'_{2r} - \frac{1}{k_0k_3q^{2r}}F^+u'_{2r-2},$$

and for $1 \leq r \leq n/2$

$$\mathbb{B}F^-u'_{2r} = \left(k_0k_3q^{2r} + \frac{1}{k_0k_3q^{2r}}\right)F^-u'_{2r} - \frac{1}{k_0k_3q^{2r}}F^-u'_{2r-2}.$$

Lemma 19.2 Assume that \mathcal{V} has X -type DDa. Then for $0 \leq r \leq (n-1)/2$

$$\begin{aligned}\mathbb{B}F^+u'_{2r} &= \left(k_0k_3q^{2r} + \frac{1}{k_0k_3q^{2r}}\right)F^+u'_{2r} - \frac{1}{k_0k_3q^{2r}}F^+u'_{2r-2}, \\ \mathbb{B}F^+u'_n &= \left(k_0k_3q^{n+1} + \frac{1}{k_0k_3q^{n+1}}\right)F^+u'_n - \frac{1-q^{n+1}}{k_0k_3q^{n+1}}F^+u'_{n-1}\end{aligned}$$

and for $1 \leq r \leq (n-1)/2$

$$\mathbb{B}F^-u'_{2r} = \left(k_0k_3q^{2r} + \frac{1}{k_0k_3q^{2r}}\right)F^-u'_{2r} - \frac{1}{k_0k_3q^{2r}}F^-u'_{2r-2}$$

Lemma 19.3 Assume that \mathcal{V} has X -type DDb. Then for $0 \leq r \leq (n-1)/2$

$$\begin{aligned}\mathbb{B}F^+u'_{2r} &= \left(k_0k_3q^{2r} + \frac{1}{k_0k_3q^{2r}}\right)F^+u'_{2r} - \frac{1}{k_0^3k_3^3q^{6r-2}}F^+u'_{2r-2}, \\ \mathbb{B}F^-u'_{2r+1} &= \left(k_0k_3q^{2r+2} + \frac{1}{k_0k_3q^{2r+2}}\right)F^-u'_{2r+1} - \frac{1}{k_0^3k_3^3q^{6r+2}}F^-u'_{2r-1}.\end{aligned}$$

Lemma 19.4 Assume that \mathcal{V} has X -type SSa. Then for $0 \leq r \leq (n-1)/2$

$$\begin{aligned}\mathbb{B}F^+u'_{2r} &= \left(k_1k_2q^{2r+1} + \frac{1}{k_1k_2q^{2r+1}}\right)F^+u'_{2r} - \frac{1}{k_1^3k_2^3q^{6r-1}}F^+u'_{2r-2}, \\ \mathbb{B}F^-u'_{2r+1} &= \left(k_1k_2q^{2r+1} + \frac{1}{k_1k_2q^{2r+1}}\right)F^-u'_{2r+1} - \frac{1}{k_1^3k_2^3q^{6r+1}}F^-u'_{2r-1}.\end{aligned}$$

Lemma 19.5 Assume that \mathcal{V} has X -type SSb. Then for $0 \leq r \leq (n-1)/2$

$$\begin{aligned}\mathbb{B}F^+u'_{2r} &= \left(k_1k_2q^{2r+1} + \frac{1}{k_1k_2q^{2r+1}}\right)F^+u'_{2r} - \frac{1-k_0^{-1}k_1k_2k_3q^{2r+1}}{k_1k_2q^{2r+1}(1-k_0^{-1}k_1k_2k_3q^{2r-1})}F^+u'_{2r-2}, \\ \mathbb{B}F^-u'_{2r} &= \left(k_1k_2q^{2r+1} + \frac{1}{k_1k_2q^{2r+1}}\right)F^-u'_{2r} - \frac{1-k_0k_1k_2k_3q^{2r+1}}{k_1k_2q^{2r+1}(1-k_0k_1k_2k_3q^{2r-1})}F^-u'_{2r-2}.\end{aligned}$$

By Lemmas 19.1–19.5 we obtain the following corollary.

Corollary 19.6 *With reference to Notation 10.9, the following hold.*

- (i) *Consider the basis for $\mathcal{V}(k_0)$ from (86). With respect to this basis the matrix representing \mathbb{B} is upper bidiagonal with the following (r, r) -entry for $0 \leq r \leq d$:*

X -type of \mathcal{V}	(r, r) -entry for \mathbb{B}	
DS, DDa, DDb	$k_0 k_3 q^{2r} + \frac{1}{k_0 k_3 q^{2r}}$	
SSa, SSb	$k_1 k_2 q^{2r+1} + \frac{1}{k_1 k_2 q^{2r+1}}$	(89)

- (ii) *Consider the basis for $\mathcal{V}(k_0^{-1})$ from (86). With respect to this basis the matrix representing \mathbb{B} is upper bidiagonal with the following (r, r) -entry for $0 \leq r \leq d'$:*

X -type of \mathcal{V}	(r, r) -entry for \mathbb{B}	
DS, DDa, DDb	$k_0 k_3 q^{2r+2} + \frac{1}{k_0 k_3 q^{2r+2}}$	
SSa, SSb	$k_1 k_2 q^{2r+1} + \frac{1}{k_1 k_2 q^{2r+1}}$	(90)

20 The equitable Askey-Wilson relations

In this section we recall some relations concerning a Leonard pair of q -Racah type.

Lemma 20.1 (See [3, Theorem 10.1].) *Let V denote a vector space over \mathbb{F} with dimension $d+1$, $d \geq 0$. Let A, A^* denote a Leonard pair on V that has q -Racah type. Let (a, b, c, d) denote a Huang data of A, A^* . Then there exists a unique \mathbb{F} -linear transformation $A^\varepsilon : V \rightarrow V$ such that*

$$A + \frac{qA^*A^\varepsilon - q^{-1}A^\varepsilon A^*}{q^2 - q^{-2}} = \frac{(q^{d+1} + q^{-d-1})(a + a^{-1}) + (b + b^{-1})(c + c^{-1})}{q + q^{-1}}I, \quad (91)$$

$$A^* + \frac{qA^\varepsilon A - q^{-1}AA^\varepsilon}{q^2 - q^{-2}} = \frac{(q^{d+1} + q^{-d-1})(b + b^{-1}) + (c + c^{-1})(a + a^{-1})}{q + q^{-1}}I, \quad (92)$$

$$A^\varepsilon + \frac{qAA^* - q^{-1}A^*A}{q^2 - q^{-2}} = \frac{(q^{d+1} + q^{-d-1})(c + c^{-1}) + (a + a^{-1})(b + b^{-1})}{q + q^{-1}}I. \quad (93)$$

The relations (91)–(93) are called the *equitable Askey-Wilson relations*.

Note 20.2 In (91)–(93) the right-hand side is invariant when we replace any of a, b, c by its inverse. Therefore A^ε does not depend on the choice of Huang data.

Lemma 20.3 *With reference to Lemma 20.1, assume $d \geq 1$. Then A^ε is the unique \mathbb{F} -linear transformation such that each of the following is a scalar multiple of the identity:*

$$A + \frac{qA^*A^\varepsilon - q^{-1}A^\varepsilon A^*}{q^2 - q^{-2}}, \quad A^* + \frac{qA^\varepsilon A - q^{-1}AA^\varepsilon}{q^2 - q^{-2}}, \quad A^\varepsilon + \frac{qAA^* - q^{-1}A^*A}{q^2 - q^{-2}}. \quad (94)$$

Proof. Let $\alpha, \beta, \gamma \in \mathbb{F}$ denote the scalars on the right in (91), (92), (93), respectively. Assume that there exists an \mathbb{F} -linear transformation $\tilde{A}^\varepsilon : V \rightarrow V$ and scalars $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in \mathbb{F}$

such that

$$A + \frac{qA^*\tilde{A}^\varepsilon - q^{-1}\tilde{A}^\varepsilon A^*}{q^2 - q^{-2}} = \tilde{\alpha}I, \quad (95)$$

$$A^* + \frac{q\tilde{A}^\varepsilon A - q^{-1}A\tilde{A}^\varepsilon}{q^2 - q^{-2}} = \tilde{\beta}I, \quad (96)$$

$$\tilde{A}^\varepsilon + \frac{qAA^* - q^{-1}A^*A}{q^2 - q^{-2}} = \tilde{\gamma}I. \quad (97)$$

We show that $A^\varepsilon = \tilde{A}^\varepsilon$. By (93) and (97)

$$\tilde{A}^\varepsilon = A^\varepsilon + (\tilde{\gamma} - \gamma)I. \quad (98)$$

In (95) eliminate \tilde{A}^ε using this, and simplify the result to find

$$A + \frac{qA^*A^\varepsilon - q^{-1}A^\varepsilon A^*}{q^2 - q^{-2}} + \frac{(\tilde{\gamma} - \gamma)A^*}{q + q^{-1}} = \tilde{\alpha}I.$$

By this and (91)

$$\alpha I + \frac{(\tilde{\gamma} - \gamma)A^*}{q + q^{-1}} = \tilde{\alpha}I,$$

and this becomes

$$(\tilde{\gamma} - \gamma)A^* = (q + q^{-1})(\tilde{\alpha} - \alpha)I.$$

This forces $\tilde{\gamma} = \gamma$ by our assumption $d \geq 1$. Now $A^\varepsilon = \tilde{A}^\varepsilon$ follows from (98). The result follows. \square

21 Huang data for the Leonard pairs obtained from a feasible \hat{H}_q -module

Throughout this section Notation 10.1 is in effect. Assume that \mathcal{V} is feasible. By Theorem 1.7 we have a Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ (resp. $\mathcal{V}(k_0^{-1})$) that has q -Racah type. Our target in this section is to prove the following result.

Proposition 21.1 *The following hold.*

(i) *The Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ has a Huang data (a, b, c, d) such that*

X -type of \mathcal{V}	a	b	c	d
DS	$k_0 k_1 q^{n/2}$	$k_0 k_3 q^{n/2}$	$k_0 k_2 q^{n/2}$	$n/2$
DDa	k_1	k_3	k_2	$(n+1)/2$
DDb	k_2	$k_0 q^{-1}$	k_1	$(n-1)/2$
SSa	$k_0 q^{-1}$	k_2	k_3	$(n-1)/2$
SSb	k_3	k_1	$k_0 q^{-1}$	$(n-1)/2$

(99)

(ii) *The Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0^{-1})$ has a Huang data (a', b', c', d') such that*

X -type of \mathcal{V}	a'	b'	c'	d'
DS	$k_0 k_1 q^{(n+2)/2}$	$k_0 k_3 q^{(n+2)/2}$	$k_0 k_2 q^{(n+2)/2}$	$(n-2)/2$
DDa	k_1	k_3	k_2	$(n-3)/2$
DDb	k_2	$k_0 q$	k_1	$(n-1)/2$
SSa	$k_0 q$	k_2	k_3	$(n-1)/2$
SSb	k_3	k_1	$k_0 q$	$(n-1)/2$

(100)

Lemma 21.2 *Define $d = \dim \mathcal{V}(k_0) - 1$ and $d' = \dim \mathcal{V}(k_0^{-1}) - 1$.*

(i) *Consider the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$. Define scalars $\{\theta_r\}_{r=0}^d$ as follows:*

X -type of \mathcal{V}	Definition of θ_r
DS, DDa, SSa	$k_0 k_1 q^{2r} + \frac{1}{k_0 k_1 q^{2r}}$
DDb, SSb	$k_2 k_3 q^{2r+1} + \frac{1}{k_2 k_3 q^{2r+1}}$

(101)

Then $\{\theta_r\}_{r=0}^d$ is a standard ordering of the eigenvalues of \mathbb{A} .

(i) *Consider the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0^{-1})$. Define scalars $\{\theta'_r\}_{r=0}^d$ as follows:*

X -type of \mathcal{V}	Definition of θ'_r
DS, DDa	$k_0 k_1 q^{2r+2} + \frac{1}{k_0 k_1 q^{2r+2}}$
DDb, SSb	$k_2 k_3 q^{2r+1} + \frac{1}{k_2 k_3 q^{2r+1}}$
SSa	$k_0 k_1 q^{2r} + \frac{1}{k_0 k_1 q^{2r}}$

(102)

Then $\{\theta'_r\}_{r=0}^d$ is a standard ordering of the eigenvalues of \mathbb{A} .

Proof. (i): By Corollaries 18.6 and 19.6 there exists a basis for $\mathcal{V}(k_0)$ with respect to which the matrix representing \mathbb{A} is lower bidiagonal and the matrix representing \mathbb{B} is upper bidiagonal. Moreover, the diagonal entries of the matrix representing \mathbb{A} are given in (87). Now the result follows by Lemma 2.4.

(ii): Similar. □

Lemma 21.3 Define $d = \dim \mathcal{V}(k_0) - 1$ and $d' = \dim \mathcal{V}(k_0^{-1}) - 1$.

(i) Consider the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$. Define scalars $\{\theta_r^*\}_{r=0}^d$ as follows:

X-type of \mathcal{V}	Definition of θ_r^*
DS, DDa, DDb	$k_0 k_3 q^{2r} + \frac{1}{k_0 k_3 q^{2r}}$
SSa, SSb	$k_1 k_2 q^{2r+1} + \frac{1}{k_1 k_2 q^{2r+1}}$

(103)

Then $\{\theta_r^*\}_{r=0}^d$ is a standard ordering of the eigenvalues of \mathbb{B} .

(i) Consider the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0^{-1})$. Define scalars $\{\theta_r^{*'}\}_{r=0}^d$ as follows:

X-type of \mathcal{V}	Definition of $\theta_r^{*'}$
DS, DDa, DDb	$k_0 k_3 q^{2r+2} + \frac{1}{k_0 k_3 q^{2r+2}}$
SSa, SSb	$k_1 k_2 q^{2r+1} + \frac{1}{k_1 k_2 q^{2r+1}}$

(104)

Then $\{\theta_r^{*'}\}_{r=0}^d$ is a standard ordering of the eigenvalues of \mathbb{B} .

Proof. Similar to the proof of Lemma 21.2. □

Lemma 21.4 (See [16, Proposition 6.6].) Define $\mathbb{C} = t_0 t_2 + (t_0 t_2)^{-1}$. Then the elements $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are related as follows:

$$\mathbb{A} + \frac{q\mathbb{B}\mathbb{C} - q^{-1}\mathbb{C}\mathbb{B}}{q^2 - q^{-2}} = \frac{(q^{-1}t_0 + qt_0^{-1})T_1 + T_2 T_3}{q + q^{-1}}, \quad (105)$$

$$\mathbb{B} + \frac{q\mathbb{C}\mathbb{A} - q^{-1}\mathbb{A}\mathbb{C}}{q^2 - q^{-2}} = \frac{(q^{-1}t_0 + qt_0^{-1})T_3 + T_1 T_2}{q + q^{-1}}, \quad (106)$$

$$\mathbb{C} + \frac{q\mathbb{A}\mathbb{B} - q^{-1}\mathbb{B}\mathbb{A}}{q^2 - q^{-2}} = \frac{(q^{-1}t_0 + qt_0^{-1})T_2 + T_3 T_1}{q + q^{-1}}. \quad (107)$$

Lemma 21.5 The following hold.

(i) Let (a, b, c, d) denote a Huang data for the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$. Assume $d \geq 1$. Then

$$c + c^{-1} = \frac{(q^{-1}k_0 + qk_0^{-1})(k_2 + k_2^{-1}) + (k_1 + k_1^{-1})(k_3 + k_3^{-1}) - (a + a^{-1})(b + b^{-1})}{q^{d+1} + q^{-d-1}}.$$

(i) Let (a', b', c', d') denote a Huang data for the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0^{-1})$. Assume $d' \geq 1$. Then

$$c' + c'^{-1} = \frac{(qk_0 + q^{-1}k_0^{-1})(k_2 + k_2^{-1}) + (k_1 + k_1^{-1})(k_3 + k_3^{-1}) - (a' + a'^{-1})(b' + b'^{-1})}{q^{d'+1} + q^{-d'-1}}.$$

Proof. (i): Note that t_0 acts on $\mathcal{V}(k_0)$ as k_0 times the identity. By this and (4), in each (105)–(107) the right-hand side acts on $\mathcal{V}(k_0)$ as a scalar multiple of the identity. By this

and Lemmas 20.1, 20.3 the action of the right-hand side of (107) is equal to the action of the right-hand side of (93). From this we obtain the result.

(ii): Similar. \square

Proof of Proposition 21.1. (i): First assume that \mathcal{V} has X -type DS. For notational convenience, Set $V = \mathcal{V}(k_0)$, and let $A : V \rightarrow V$ (resp. $A^* : V \rightarrow V$) denote the \mathbb{F} -linear transformation that is induced by the action of \mathbb{A} (resp. \mathbb{B}). By Corollary 10.11 the Leonard pair A, A^* has diameter $d = n/2$. Define scalars $\{\theta_r\}_{r=0}^d$ and $\{\theta_r^*\}_{r=0}^d$ as

$$\theta_r = k_0 k_1 q^{2r} + \frac{1}{k_0 k_1 q^{2r}}, \quad \theta_r^* = k_0 k_3 q^{2r} + \frac{1}{k_0 k_3 q^{2r}} \quad (0 \leq r \leq d). \quad (108)$$

By Lemmas 21.2 (resp. Lemma 21.3) the sequence $\{\theta_r\}_{r=0}^d$ (resp. $\{\theta_r^*\}_{r=0}^d$) is a standard ordering of the eigenvalues of A (resp. A^*). Now define scalars

$$a = k_0 k_1 q^d, \quad b = k_0 k_3 q^d, \quad c = k_0 k_2 q^d.$$

By (108)

$$\theta_r = a q^{2r-d} + a^{-1} q^{d-2r}, \quad \theta_r^* = b q^{2r-d} + b^{-1} q^{d-2r} \quad (0 \leq r \leq d).$$

By Lemma 12.1 $k_0 k_1 k_2 k_3 = q^{-2d-1}$. Using this, one checks that a, b, c satisfy the displayed equation in Lemma 21.5(i). By this and the last sentence in Lemma 2.5, (a, b, c, d) is a Huang data of A, A^* . We have shown the result when \mathcal{V} has X -type DS. The proof is similar for the other types.

(ii): Similar. \square

Corollary 21.6 *The following hold.*

(i) *For the Huang data (a, b, c, d) from Proposition 21.1(i) the parameters $\{k_i\}_{i \in \mathbb{I}}$ satisfy*

Case	k_0	k_1	k_2	k_3
DS	$(abcq^{1-d})^{1/2}$	$aq^{-d}k_0^{-1}$	$cq^{-d}k_0^{-1}$	$bq^{-d}k_0^{-1}$
DDa	q^{-d}	a	c	b
DDb	bq	c	a	q^{-d-1}
SSa	aq	q^{-d-1}	b	c
SSb	cq	b	q^{-d-1}	a

(ii) *For the Huang data (a', b', c', d') from Proposition 21.1(ii) the parameters $\{k_i\}_{i \in \mathbb{I}}$ satisfy*

Case	k_0	k_1	k_2	k_3
DS	$(a'b'c'q^{-d'-3})^{1/2}$	$a'q^{-d'-2}k_0^{-1}$	$c'q^{-d'-2}k_0^{-1}$	$b'q^{-d'-2}k_0^{-1}$
DDa	$q^{-d'-2}$	a'	c'	b'
DDb	$b'q^{-1}$	c'	a'	$q^{-d'-1}$
SSa	$a'q^{-1}$	$q^{-d'-1}$	b'	c'
SSb	$c'q^{-1}$	b'	$q^{-d'-1}$	a'

Proof. Use Lemma 12.1. \square

Note 21.7 In Corollary 21.6(i) the assertion $k_0 = (abcq^{1-d})^{1/2}$ means that k_0 is equal to one of the square roots of $abcq^{1-d}$. Similar for (ii).

22 Proof of Theorem 1.8; “only if” direction

In this section we prove the “only if” direction of Theorem 1.8.

Lemma 22.1 *With reference to Notation 10.1, assume that \mathcal{V} is feasible and has X -type DS. Define scalars a, b, c, d as in (99). Then $a^2 \neq q^{-2d}$ and $b^2 \neq q^{-2d}$.*

Proof. By Corollary 14.8 $k_0^2 k_1^2 \neq q^{-2n}$. By Lemma 12.4 $k_0^2 k_3^2 \neq q^{-2n}$. The result follows from these comments and (99). \square

Lemma 22.2 *With reference to Notation 10.1, assume that \mathcal{V} is feasible and has X -type among DDb, SSa, SSb. Define scalars a, b, c, d as in (99). Then the following inequalities hold:*

X -type of \mathcal{V}	Inequalities		
DDb	$b^2 \neq q^{-2}$	$a^2 \neq q^{\pm 2d}$	
SSa	$a^2 \neq q^{-2}$	$b^2 \neq q^{\pm 2d}$	
SSb	$c^2 \neq q^{-2}$	$a^2 \neq q^{\pm 2d}$	$b^2 \neq q^{\pm 2d}$

Proof. First assume that \mathcal{V} has X -type DDb. We have $k_0^2 \neq 1$ since \mathcal{V} is feasible. By Lemma 12.1 $k_3^2 = q^{-n-1}$. By Corollary 14.8 $k_2^2 k_3^2$ is not among q^{-2}, q^{-2n} . The result follows from these comments. Next assume that \mathcal{V} has X -type SSa. By Lemma 12.1 $k_1^2 = q^{-n-1}$. By Corollary 14.8 $k_0^2 k_1^2 \neq q^{-n-1}$. By Lemma 12.4 k_2^2 is not among q^{n-1}, q^{1-n} . The result follows from these comments. Next assume that \mathcal{V} has X -type SSb. We have $k_0^2 \neq 1$ since \mathcal{V} is feasible. By Lemma 12.1 $k_2^2 = q^{-n-1}$. By Corollary 14.8 $k_2^2 k_3^2$ is not among q^{-2}, q^{-2n} . By Lemma 12.4 k_1^2 is not among q^{n-1}, q^{1-n} . The result follows from these comments. \square

Corollary 22.3 *With reference to Notation 10.1, assume that \mathcal{V} is feasible. Then there exist a Huang data (a, b, c, d) of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ and a Huang data (a', b', c', d') of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0^{-1})$ such that the following hold:*

X -type of \mathcal{V}	$d' - d$	a'/a	b'/b	c'/c	Inequalities		
DS	-1	q	q	q	$a^2 \neq q^{-2d}$	$b^2 \neq q^{-2d}$	
DDa	-2	1	1	1			
DDb	0	1	q^2	1	$a^2 \neq q^{\pm 2d}$		$b^2 \neq q^{-2}$
SSa	0	q^2	1	1		$b^2 \neq q^{\pm 2d}$	$a^2 \neq q^{-2}$
SSb	0	1	1	q^2	$a^2 \neq q^{\pm 2d}$	$b^2 \neq q^{\pm 2d}$	$c^2 \neq q^{-2}$

(109)

Proof. Immediate from Proposition 21.1 and Lemmas 22.1, 22.2. \square

Proof of Theorem 1.8; “only if” direction. Let A, A^* (resp. A', A'^*) denote a Leonard pair on V (resp. V') that has q -Racah type. Assume that these Leonard pairs are linked, so there exists a feasible \hat{H}_q -module structure on $\mathcal{V} := V \oplus V'$ such that V, V' are the eigenspaces of t_0 and (6) holds. Let $\{k_i\}_{i \in \mathbb{I}}$ denote a parameter sequence of \mathcal{V} that is consistent with a standard ordering of the eigenvalues of X . First assume $V = \mathcal{V}(k_0)$ and $V' = \mathcal{V}(k_0^{-1})$. By Corollary 22.3 there exist a Huang data (a, b, c, d) of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ and a Huang data (a', b', c', d') of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0^{-1})$ that satisfy (109). Note by (6) that (a, b, c, d) is a Huang data of A, A^* and (a', b', c', d') is a Huang data of A', A'^* . By (109), for each X -type of \mathcal{V} , these Huang data satisfy the following case in Theorem 1.8:

X -type of \mathcal{V}	DS	DDa	DDb	SSa	SSb
Case	(ii)	(i)	(iv)	(iii)	(v)

Next assume $V = \mathcal{V}(k_0^{-1})$ and $V' = \mathcal{V}(k_0)$. By Corollary 22.3 in which (a, b, c, d) and (a', b', c', d') are exchanged, for each X -type of \mathcal{V} there exist a Huang data (a', b', c', d') of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ and a Huang data (a, b, c, d) of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0^{-1})$ that satisfy

X -type of \mathcal{V}	$d' - d$	a'/a	b'/b	c'/c	Inequalities		
DS	1	q^{-1}	q^{-1}	q^{-1}	$a'^2 \neq q^{-2d'}$	$b'^2 \neq q^{-2d'}$	
DDa	2	1	1	1			
DDb	0	1	q^{-2}	1	$a'^2 \neq q^{\pm 2d'}$		$b'^2 \neq q^{-2}$
SSa	0	q^{-2}	1	1		$b'^2 \neq q^{\pm 2d'}$	$a'^2 \neq q^{-2}$
SSb	0	1	1	q^{-2}	$a'^2 \neq q^{\pm 2d'}$	$b'^2 \neq q^{\pm 2d'}$	$c'^2 \neq q^{-2}$

In this table we rewrite the inequalities in terms of a, b, c, d :

X -type of \mathcal{V}	$d' - d$	a'/a	b'/b	c'/c	Inequalities		
DS	1	q^{-1}	q^{-1}	q^{-1}	$a^2 \neq q^{-2d}$	$b^2 \neq q^{-2d}$	
DDa	2	1	1	1			
DDb	0	1	q^{-2}	1	$a^2 \neq q^{\pm 2d}$		$b^2 \neq q^2$
SSa	0	q^{-2}	1	1		$b^2 \neq q^{\pm 2d}$	$a^2 \neq q^2$
SSb	0	1	1	q^{-2}	$a^2 \neq q^{\pm 2d}$	$b^2 \neq q^{\pm 2d}$	$c^2 \neq q^2$

Thus case (vi) holds if \mathcal{V} has X -type DS, and case (vii) holds if \mathcal{V} has X -type DDa. If \mathcal{V} has X -type among DDb, SSa, SSb, we replace each of a, b, c, a', b', c' with its inverse. This gives

X -type of \mathcal{V}	$d' - d$	a'/a	b'/b	c'/c	Inequalities		
DDb	0	1	q^2	1	$a^{-2} \neq q^{\pm 2d}$		$b^{-2} \neq q^2$
SSa	0	q^2	1	1		$b^{-2} \neq q^{\pm 2d}$	$a^{-2} \neq q^2$
SSb	0	1	1	q^2	$a^{-2} \neq q^{\pm 2d}$	$b^{-2} \neq q^{\pm 2d}$	$c^{-2} \neq q^2$

For each X -type among DDb, SSa, SSb these Huang data satisfy the following case in Theorem 1.8:

X -type of \mathcal{V}	DDb	SSa	SSb
Case	(iv)	(iii)	(v)

The result follows. \square

23 Construction of an XD \hat{H}_q -module

With reference to Notation 10.1, let T denote the X -type of \mathcal{V} . By Lemmas 12.1 and 12.4 the parameters $\{k_i\}_{i \in \mathbb{I}}$ satisfy the following conditions:

T	Conditions	
DS	$k_0 k_1 k_2 k_3 = q^{-n-1}$	(110)
	Neither of $\pm k_0 k_3$ is among $q^{-1}, q^{-2}, q^{-3}, \dots, q^{-n}$	
	None of $\pm k_0, \pm k_1, \pm k_2, \pm k_3$ is among $q^{-1}, q^{-2}, q^{-3}, \dots, q^{-n/2}$	
DDa	$k_0^2 = q^{-n-1}$	
	None of $\pm k_3^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$	
	None of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$	
DDb	$k_3^2 = q^{-n-1}$	
	None of $\pm k_0^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$	
	None of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$	
SSa	$k_1^2 = q^{-n-1}$	(110)
	None of $\pm k_2^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$	
	None of $k_1 k_2 k_0^{\pm 1} k_3^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$	
SSb	$k_2^2 = q^{-n-1}$	
	None of $\pm k_1^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$	
	None of $k_1 k_2 k_0^{\pm 1} k_3^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$	

In this section we prove the following result.

Proposition 23.1 *Let $n \geq 0$ denote an integer and let $\{k_i\}_{i \in \mathbb{I}}$ denote nonzero scalars in \mathbb{F} . Let T be among DS, DDa, DDb, SSa, SSb. Assume that n is even if T is DS and odd otherwise. Assume that $\{k_i\}_{i \in \mathbb{I}}$ satisfy the conditions (110). Then there exists an XD \hat{H}_q -module \mathcal{V} with dimension $n+1$ that has X -type T and parameter sequence $\{k_i\}_{i \in \mathbb{I}}$ that is consistent with a standard ordering of the eigenvalues of X .*

Let $n \geq 0$ denote an integer and let $\{k_i\}_{i \in \mathbb{I}}$ denote nonzero scalars in \mathbb{F} . Fix T among DS, DDa, DDb, SSa, SSb. Assume that n is even if T is DS and odd otherwise. Assume that $\{k_i\}_{i \in \mathbb{I}}$ satisfy the conditions (110). Define scalars $\{\mu_r\}_{r=0}^n$ as follows. If T is among DS, DDa, DDb,

$$\mu_r = \begin{cases} k_0 k_3 q^r & \text{if } r \text{ is even,} \\ \frac{1}{k_0 k_3 q^{r+1}} & \text{if } r \text{ is odd} \end{cases} \quad (0 \leq r \leq n). \quad (111)$$

If T is among SSa, SSb,

$$\mu_r = \begin{cases} \frac{1}{k_1 k_2 q^{r+1}} & \text{if } r \text{ is even,} \\ k_1 k_2 q^r & \text{if } r \text{ is odd} \end{cases} \quad (0 \leq r \leq n). \quad (112)$$

Lemma 23.2 *The scalars $\{\mu_r\}_{r=0}^n$ are mutually distinct.*

Proof. Routine using (110). \square

Lemma 23.3 *The reduced diagram of $\{\mu_r\}_{r=0}^n$ is as follows:*

T	Diagram	
DS	$\mu_0 \quad \mu_1 \quad \mu_2 \quad \mu_3 \quad \dots \quad \mu_{n-1} \quad \mu_n$ 	(113)
DDa, DDb	$\mu_0 \quad \mu_1 \quad \mu_2 \quad \mu_3 \quad \dots \quad \mu_{n-1} \quad \mu_n$ 	
SSa, SSb	$\mu_0 \quad \mu_1 \quad \mu_2 \quad \mu_3 \quad \dots \quad \mu_{n-1} \quad \mu_n$ 	

Proof. Follows from (111) and (112). \square

We now construct an \hat{H}_q -module. Let \mathcal{V} denote a vector space over \mathbb{F} with dimension $n + 1$, and let $\{v_r\}_{r=0}^n$ denote a basis for \mathcal{V} . We define the action of $\{t_i\}_{i \in \mathbb{I}}$ on $\{v_r\}_{r=0}^n$ as follows. Recall the function G from (31). For $0 \leq r \leq n - 1$ such that μ_r, μ_{r+1} are 1-adjacent, we define the action of t_0, t_3 by

$$t_0 v_r = \frac{\mu_r(k_0 + k_0^{-1}) - k_3 - k_3^{-1}}{\mu_r - \mu_r^{-1}} v_r + \frac{\mu_r}{\mu_r - \mu_r^{-1}} v_{r+1}, \quad (114)$$

$$t_0 v_{r+1} = \frac{G(\mu_r, k_0, k_3)}{\mu_r(\mu_r^{-1} - \mu_r)} v_r + \frac{\mu_r^{-1}(k_0 + k_0^{-1}) - k_3 - k_3^{-1}}{\mu_r^{-1} - \mu_r} v_{r+1}, \quad (115)$$

$$t_3 v_r = \frac{\mu_r(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\mu_r - \mu_r^{-1}} v_r + \frac{1}{\mu_r^{-1} - \mu_r} v_{r+1}, \quad (116)$$

$$t_3 v_{r+1} = \frac{G(\mu_r, k_0, k_3)}{\mu_r - \mu_r^{-1}} v_r + \frac{\mu_r^{-1}(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\mu_r^{-1} - \mu_r} v_{r+1}. \quad (117)$$

For $0 \leq r \leq n - 1$ such that μ_r, μ_{r+1} are q -adjacent, we define the action of t_1, t_2 by

$$t_1 v_r = \frac{q^{-1}\mu_r^{-1}(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\mu_r^{-1} - q\mu_r} v_r + \frac{1}{q\mu_r - q^{-1}\mu_r^{-1}} v_{r+1}, \quad (118)$$

$$t_1 v_{r+1} = \frac{G(q\mu_r, k_1, k_2)}{q^{-1}\mu_r^{-1} - q\mu_r} v_r + \frac{q\mu_r(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{q\mu_r - q^{-1}\mu_r^{-1}} v_{r+1}, \quad (119)$$

$$t_2 v_r = \frac{q^{-1}\mu_r^{-1}(k_2 + k_2^{-1}) - k_1 - k_1^{-1}}{q^{-1}\mu_r^{-1} - q\mu_r} v_r + \frac{q^{-1}\mu_r^{-1}}{q^{-1}\mu_r^{-1} - q\mu_r} v_{r+1}, \quad (120)$$

$$t_2 v_{r+1} = \frac{q\mu_r G(q\mu_r, k_1, k_2)}{q\mu_r - q^{-1}\mu_r^{-1}} v_r + \frac{q\mu_r(k_2 + k_2^{-1}) - k_1 - k_1^{-1}}{q\mu_r - q^{-1}\mu_r^{-1}} v_{r+1}. \quad (121)$$

In (114)–(121) the denominators are nonzero by Lemmas 23.2 and 23.3. We have defined some actions of $\{t_i\}_{i \in \mathbb{I}}$. The remaining actions are defined as follows:

T	Action			
DS	$t_0 v_0 = k_0 v_0$	$t_3 v_0 = k_3 v_0$	$t_1 v_n = k_1 v_n$	$t_2 v_n = k_2 v_n$
DDa	$t_0 v_0 = k_0 v_0$	$t_3 v_0 = k_3 v_0$	$t_0 v_n = k_0 v_n$	$t_3 v_n = k_3^{-1} v_n$
DDb	$t_0 v_0 = k_0 v_0$	$t_3 v_0 = k_3 v_0$	$t_0 v_n = k_0^{-1} v_n$	$t_3 v_n = k_3 v_n$
SSa	$t_1 v_0 = k_1 v_0$	$t_2 v_0 = k_2 v_0$	$t_1 v_n = k_1 v_n$	$t_2 v_n = k_2^{-1} v_n$
SSb	$t_1 v_0 = k_1 v_0$	$t_2 v_0 = k_2 v_0$	$t_1 v_n = k_1^{-1} v_n$	$t_2 v_n = k_2 v_n$

(122)

For $i \in \mathbb{I}$ we define the action of t_i^{-1} on $\{v_r\}_{r=0}^n$ by

$$t_i^{-1} v_r = (k_i + k_i^{-1}) v_r - t_i v_r \quad (0 \leq r \leq n). \quad (123)$$

We have defined the action of $\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$ on $\{v_r\}_{r=0}^n$.

Lemma 23.4 *The above actions of $\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$ on $\{v_r\}_{r=0}^n$ give an \hat{H}_q -module structure on \mathcal{V} .*

Proof. One routinely checks that the defining relations (1)–(3) of \hat{H}_q hold on $\{v_r\}_{r=0}^n$. \square

Lemma 23.5 *For $0 \leq r \leq n$ the vector v_r is an eigenvector of X with eigenvalue μ_r . Moreover X is diagonalizable on \mathcal{V} .*

Proof. Pick any integer r such that $0 \leq r \leq n$. One checks $t_3 t_0 v_r = \mu_r v_r$. Therefore v_r is an eigenvector of X with eigenvalue μ_r . Now X has $n+1$ mutually distinct eigenvalues on \mathcal{V} by Lemma 23.2. So X is diagonalizable on \mathcal{V} . \square

Lemma 23.6 *For $0 \leq r \leq n-1$ the following hold.*

- (i) *Assume that μ_r, μ_{r+1} are 1-adjacent. Then $G_0 v_r = v_{r+1}$ and $G_0 v_{r+1} = G(\mu_r, k_0, k_3) v_r$.*
- (ii) *Assume that μ_r, μ_{r+1} are q -adjacent. Then $G_2 v_r = v_{r+1}$ and $G_2 v_{r+1} = G(q\mu_r, k_1, k_2) v_r$.*

Proof. Routine verification. \square

Lemma 23.7 *For $0 \leq r \leq n-1$ the following hold.*

- (i) *Assume that μ_r, μ_{r+1} are 1-adjacent. Then $G(\mu_r, k_0, k_3) \neq 0$.*
- (ii) *Assume that μ_r, μ_{r+1} are q -adjacent. Then $G(q\mu_r, k_1, k_2) \neq 0$.*

Proof. (i): We claim that μ_r is not among $k_0 k_3, k_0 k_3^{-1}, k_0^{-1} k_3, k_0^{-1} k_3^{-1}$. First assume that \mathbb{T} is DS. Note that r is odd by (113) and since μ_r, μ_{r+1} are 1-adjacent. By (111)

$\mu_r = (k_0 k_3 q^{r+1})^{-1}$. By (110) $k_0^2 k_3^2$ is not among $q^{-2}, q^{-4}, \dots, q^{-2n}$, and neither of k_0^2, k_3^2 is among $q^{-2}, q^{-4}, \dots, q^{-n}$. By these comments μ_r is not among $k_0 k_3, k_0 k_3^{-1}, k_0^{-1} k_3$. Moreover μ_r is not equal to $k_0^{-1} k_3^{-1}$ since q is not a root of unity. Thus the claim holds when T is DS. Next assume that T is DDa. Note that n is odd by the construction, and r is odd by (113) and since μ_r, μ_{r+1} are 1-adjacent. By (111) $\mu_r = (k_0 k_3 q^{r+1})^{-1}$. By (110) $k_0^2 = q^{-n-1}$, and k_3^2 is not among $1, q^2, q^4, \dots, q^{n-1}$. By these comments μ_r is not among $k_0 k_3, k_0^{-1} k_3$. Moreover μ_r is not among $k_0 k_3^{-1}, k_0^{-1} k_3^{-1}$ since q is not a root of unity. Thus the claim holds when T is DDa. In a similar way we can show the claim when T is among DDb, SSa, SSb. Now $G(\mu_r, k_0, k_3) \neq 0$ by (31) and the claim.

(ii): Similar. \square

Lemma 23.8 *The \hat{H}_q -module \mathcal{V} is irreducible and has parameter sequence $\{k_i\}_{i \in \mathbb{I}}$.*

Proof. Let \mathcal{W} denote a nonzero \hat{H}_q -submodule of \mathcal{V} . We claim that $v_s \in \mathcal{W}$ for some s ($0 \leq s \leq n$). Let s ($0 \leq s \leq n$) denote the maximal integer such that $\mathcal{W} \cap \sum_{r=s}^n \mathbb{F} v_r$ is nonzero. Pick a nonzero vector w in $\mathcal{W} \cap \sum_{r=s}^n \mathbb{F} v_r$, and write $w = \sum_{r=s}^n \alpha_r v_r$. Note that $\alpha_s \neq 0$ by the maximality of s . By Lemma 23.5 $Xw = \sum_{r=s}^n \alpha_r \mu_r v_r$. By these comments

$$\mu_s w - Xw = \sum_{r=s+1}^n (\mu_s - \mu_r) \alpha_r v_r.$$

By the maximality of s we must have $\mu_s w - Xw = 0$. By Lemma 23.2 $\mu_s - \mu_r \neq 0$ for $s+1 \leq r \leq n$. By these comments $\alpha_r = 0$ for $s+1 \leq r \leq n$. Therefore $w = \alpha_s v_s$ and the claim follows. By the claim and Lemmas 23.6, 23.7 we find that $v_r \in \mathcal{W}$ for all r ($0 \leq r \leq n$). So $\mathcal{W} = \mathcal{V}$. We have shown that \mathcal{V} is an irreducible \hat{H}_q -module. By (123) $\{k_i\}_{i \in \mathbb{I}}$ is a parameter sequence of \mathcal{V} . \square

Proof of Proposition 23.1. In the above we have constructed an \hat{H}_q -module \mathcal{V} . By Lemmas 23.5 and 23.8 the \hat{H}_q -module \mathcal{V} is XD, and $\{k_i\}_{i \in \mathbb{I}}$ is a parameter sequence of \mathcal{V} . By (122) and Lemmas 23.3, 23.5, \mathcal{V} has X -type T and $\{\mu_r\}_{r=0}^n$ is a standard ordering of the eigenvalues of X . Moreover the parameter sequence $\{k_i\}_{i \in \mathbb{I}}$ is consistent with the ordering $\{\mu_r\}_{r=0}^n$. The result follows. \square

24 Proof of Theorem 1.8; “if” direction

In this section we prove the “if” direction of Theorem 1.8. Let A, A^* denote a Leonard pair on V and let A', A'^* denote a Leonard pair on V' . Assume that these Leonard pairs have q -Racah type, and let (a, b, c, d) (resp. (a', b', c', d')) denote a Huang data of A, A^* (resp. A', A'^*). Assume that these Huang data satisfy one of the conditions (i)–(vii) in Theorem 1.8. We show that there exists a feasible \hat{H}_q -module structure on $V \oplus V'$ such that V, V' are the eigenspaces of t_0 and (6) holds. We may assume that the Huang data satisfy one of (i)–(v) by exchanging our two Leonard pairs if necessary (see Remark 1.11). For notational

convenience, we rename the cases (i)–(v) to increase the compatibility with Corollary 22.3; thus we assume that one of the following cases occurs:

Case	$d' - d$	a'/a	b'/b	c'/c	Inequalities		
DS	-1	q	q	q	$a^2 \neq q^{-2d}$	$b^2 \neq q^{-2d}$	
DDa	-2	1	1	1			
DDb	0	1	q^2	1	$a^2 \neq q^{\pm 2d}$		$b^2 \neq q^{-2}$
SSa	0	q^2	1	1		$b^2 \neq q^{\pm 2d}$	$a^2 \neq q^{-2}$
SSb	0	1	1	q^2	$a^2 \neq q^{\pm 2d}$	$b^2 \neq q^{\pm 2d}$	$c^2 \neq q^{-2}$

(124)

For each of the above cases we define $\{k_i\}_{i \in \mathbb{I}}$ as follows:

Case	k_0	k_1	k_2	k_3
DS	$(abcq^{1-d})^{1/2}$	$aq^{-d}k_0^{-1}$	$cq^{-d}k_0^{-1}$	$bq^{-d}k_0^{-1}$
DDa	q^{-d}	a	c	b
DDb	bq	c	a	q^{-d-1}
SSa	aq	q^{-d-1}	b	c
SSb	cq	b	q^{-d-1}	a

(125)

In case DS we may take either square root as the value of k_0 ; see Remark 24.7 below. Set $n = d + d' + 1$. So $V \oplus V'$ has dimension $n + 1$. Note that n is even in case DS, and odd in the other cases. The following two lemmas are immediate from (125).

Lemma 24.1 *The scalars $\{k_i\}_{i \in \mathbb{I}}$ satisfy the following equation:*

Case	Equation
DS	$k_0 k_1 k_2 k_3 = q^{-n-1}$
DDa	$k_0^2 = q^{-n-1}$
DDb	$k_3^2 = q^{-n-1}$
SSa	$k_1^2 = q^{-n-1}$
SSb	$k_2^2 = q^{-n-1}$

(126)

Lemma 24.2 *The Huang data (a, b, c, d) and (a', b', c', d') are represented in terms of*

$\{k_i\}_{i \in \mathbb{I}}$ as follows:

Case	a a'	b b'	c c'	d d'
DS	$k_0 k_1 q^{n/2}$ $k_0 k_1 q^{(n+2)/2}$	$k_0 k_3 q^{n/2}$ $k_0 k_3 q^{(n+2)/2}$	$k_0 k_2 q^{n/2}$ $k_0 k_2 q^{(n+2)/2}$	$n/2$ $(n-2)/2$
DDa	k_1 k_1	k_3 k_3	k_2 k_2	$(n+1)/2$ $(n-3)/2$
DDb	k_2 k_2	$k_0 q^{-1}$ $k_0 q$	k_1 k_1	$(n-1)/2$ $(n-1)/2$
SSa	$k_0 q^{-1}$ $k_0 q$	k_2 k_2	k_3 k_3	$(n-1)/2$ $(n-1)/2$
SSb	k_3 k_3	k_1 k_1	$k_0 q^{-1}$ $k_0 q$	$(n-1)/2$ $(n-1)/2$

(127)

By Lemma 2.8 the Huang data (a, b, c, d) and (a', b', c', d') satisfy the following inequalities:

$$\text{Neither of } a^2, b^2 \text{ is among } q^{2d-2}, q^{2d-4}, \dots, q^{2-2d}. \quad (128)$$

$$\text{None of } abc, a^{-1}bc, ab^{-1}c, abc^{-1} \text{ is among } q^{d-1}, q^{d-3}, \dots, q^{1-d}. \quad (129)$$

$$\text{Neither of } a'^2, b'^2 \text{ is among } q^{2d'-2}, q^{2d'-4}, \dots, q^{2-2d'}. \quad (130)$$

$$\text{None of } a'b'c', a'^{-1}b'c', a'b'^{-1}c', a'b'c'^{-1} \text{ is among } q^{d'-1}, q^{d'-3}, \dots, q^{1-d'}. \quad (131)$$

The inequalities (128)–(131) have the following consequence.

Lemma 24.3 *The scalars $\{k_i\}_{i \in \mathbb{I}}$ satisfy the following inequalities:*

Case	Inequalities
DS	None of $\pm k_0 k_3, \pm k_0 k_1$ is among $q^{-1}, q^{-2}, q^{-3}, \dots, q^{-n}$ None of $\pm k_0, \pm k_1, \pm k_2, \pm k_3$ is among $q^{-1}, q^{-2}, q^{-3}, \dots, q^{-n/2}$
DDa	None of $\pm k_1^{\pm 1}, \pm k_3^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$ None of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$
DDb	None of $\pm k_0^{\pm 1}, \pm k_2^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$ None of $k_0 k_3 k_1^{\pm 1} k_2^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$
SSa	None of $\pm k_0^{\pm 1}, \pm k_2^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$ None of $k_1 k_2 k_0^{\pm 1} k_3^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$
SSb	None of $\pm k_1^{\pm 1}, \pm k_3^{\pm 1}$ is among $1, q, q^2, \dots, q^{(n-1)/2}$ None of $k_1 k_2 k_0^{\pm 1} k_3^{\pm 1}$ is among $q^{-1}, q^{-3}, q^{-5}, \dots, q^{-n}$

Proof. Routine verification using (124), (125), (128)–(131). □

Let \mathbf{T} be among DS, DDa, DDb, SSa, SSb. By Lemmas 24.1, 24.3 and Proposition 23.1 there exists an XD \hat{H}_q -module \mathcal{V} with dimension $n+1$ that has X -type \mathbf{T} and parameter sequence $\{k_i\}_{i \in \mathbb{I}}$ that is consistent with a standard ordering of the eigenvalues of X .

Lemma 24.4 $k_0^2 \neq 1$.

Proof. Note that the value of k_0 is given in (125). In case DS, $k_0^2 = abcq^{1-d}$ and so $k_0^2 \neq 1$ by (129). In case DDa, $k_0^2 = q^{-2d}$ and so $k_0^2 \neq 1$ by $d = d' + 2 \geq 2$ and since q is not a root of unity. In case DDb, $k_0^2 = b^2 q^2$, so $k_0^2 \neq 1$ since $b^2 \neq q^{-2}$ by (124). The proof is similar for the cases SSa, SSb. \square

Lemma 24.5 t_0 has two distinct eigenvalues k_0, k_0^{-1} on \mathcal{V} .

Proof. Note that t_0 has two distinct eigenvalues if and only if each of $F^+\mathcal{V}$ and $F^-\mathcal{V}$ is nonzero. First assume that the reduced X -diagram of \mathcal{V} has a single bond. Then the result follows from Lemma 10.5(ii). Next assume that the reduced X -diagram has no single bond. Then $n = 1$ and the reduced X -diagram of \mathcal{V} is a double bond. So \mathcal{V} has X -type DDa or DDb. If \mathcal{V} has X -type DDa then $n \geq 3$ since $d = d' + 2 \geq 2$, contradicting $n = 1$. Thus \mathcal{V} has X -type DDb. By Lemma 8.2 the eigenvalues of t_0 on $\mathcal{V}_X(\mu_0)$ and $\mathcal{V}_X(\mu_1)$ are reciprocals. By Lemma 24.4 $k_0 \neq k_0^{-1}$. By these comments t_0 has two distinct eigenvalues on \mathcal{V} . \square

Lemma 24.6 The \hat{H}_q -module \mathcal{V} is feasible.

Proof. By the construction \mathcal{V} is XD. By Corollary 14.8 and Lemmas 24.1, 24.3, Y is diagonalizable on \mathcal{V} , so \mathcal{V} is YD. By Lemma 24.5 t_0 has two distinct eigenvalues on \mathcal{V} . Thus \mathcal{V} is feasible. \square

Proof of Theorem 1.8; “if” direction. By Theorem 1.7 and Lemma 24.6 the pair \mathbb{A}, \mathbb{B} acts on $\mathcal{V}(k_0)$ (resp. $\mathcal{V}(k_0^{-1})$) as a Leonard pair of q -Racah type. Moreover comparing (99), (100) with (127) we find that the Huang data of \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ (resp. $\mathcal{V}(k_0^{-1})$) coincides with the Huang data of A, A^* (resp. A', A'^*). By this and Lemma 2.7 the Leonard pair \mathbb{A}, \mathbb{B} on $\mathcal{V}(k_0)$ (resp. $\mathcal{V}(k_0^{-1})$) is isomorphic to the Leonard pair A, A^* on V (resp. A', A'^* on V'). Let $f : \mathcal{V}(k_0) \rightarrow V$ (resp. $f' : \mathcal{V}(k_0^{-1}) \rightarrow V'$) denote an isomorphism of Leonard pairs. Recall that $\mathcal{V} = \mathcal{V}(k_0) + \mathcal{V}(k_0^{-1})$ (direct sum). So we have the \mathbb{F} -linear bijection $f \oplus f' : \mathcal{V} \rightarrow V \oplus V'$. We define the \hat{H}_q -module structure on $V \oplus V'$ so that $f \oplus f'$ is an isomorphism of \hat{H}_q -modules. In this \hat{H}_q -module the spaces V, V' are the eigenspaces of t_0 , and (6) holds by the construction. Therefore the Leonard pairs A, A^* and A', A'^* are linked. \square

Remark 24.7 We defined the integer n and the scalars $\{k_i\}_{i \in \mathbb{I}}$ in (125), and constructed a feasible \hat{H}_q -module \mathcal{V} that has dimension $n + 1$ and parameter sequence $\{k_i\}_{i \in \mathbb{I}}$. In the definition of k_0 in case DS, there appears a square root, so the value of k_0 is determined up to sign. By our construction we obtain an \hat{H}_q -module from each of two values of k_0 . These two \hat{H}_q -modules are not isomorphic; otherwise these two \hat{H}_q -modules must have the same parameters up to reciprocal.

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